

# IMPERFECT KNOWLEDGE, ADAPTIVE LEARNING AND THE BIAS AGAINST ACTIVIST MONETARY POLICIES

by Alberto Locarno\*

## Abstract

When the economy is subject to recurrent structural shifts, the monetary authority cannot credibly commit to a systematic approach to policy, since consistency between promises and actions is not easily verifiable; moreover, since agents have incomplete knowledge of the surrounding environment, they form expectations that may deviate substantially from the full-information case. The present paper studies the implications for the effectiveness of discretionary monetary policymaking of departing from the benchmark of rational expectations and assuming instead that agents learn adaptively. It focuses on two issues, namely whether imperfect knowledge generates a bias against stabilisation policies and whether the optimal monetary strategy takes the form of an inflation cap. Rules featuring an inflation cap are not only justified on theoretical grounds, but are also appealing because they seem appropriate to deal with imperfect knowledge and learning: by setting explicit bounds on inflation, they seem better suited to restrain expectations from drifting significantly away from target, thus removing one of the main sources of policy ineffectiveness. The main findings of the paper are the following. First, when agents do not possess complete knowledge on the structure of the economy and rely on an adaptive learning technology, a bias toward conservativeness arises. Second, a policy that involves a cap on inflation is helpful in reducing output and inflation variability, but it is not uniformly superior to a strategy aimed at minimising a quadratic loss function. Third, the bias against stabilisation policies and towards conservativeness does not depend on whether agents have finite or infinite memory.

**JEL classification:** E52,E31,D84.

**KEYWORDS:** Adaptive learning, optimal degree of monetary policy discretion, bias against activist policies.

---

\* Bank of Italy, Research Department

## Contents

1. Introduction .....	9
2. Preferences and uncertainty .....	12
3. The model .....	15
3.1 The structure of the economy .....	16
3.2 Central bank loss function .....	18
3.3 Signal extraction and the rational expectations equilibrium .....	19
4. Adaptive learning and monetary policy regimes .....	24
4.1 Private sector learning .....	27
4.2 Private sector and central bank learning .....	30
5. Imperfect knowledge and policy effectiveness .....	34
6. Conclusions .....	41
Appendix .....	43
References .....	51



## 1. Introduction<sup>1</sup>

The effectiveness of policymaking depends on the monetary authority committing to a systematic approach to policy. As long as price setting depends on expectations, a central bank that can establish credibility one way or another may be able to face an improved short-run trade-off between inflation and output and thus to it may reduce inflation at lower costs. This is because the monetary authority can manipulate expectations to exploit the dependence of current inflation on future demand. However, for policymakers to succeed in steering expectations and reaping the benefits of commitment, agents must be able fully to anticipate the future impact of monetary decisions on the economy, which is feasible only if the economic environment is stable and expectations are rational. When, on the contrary, the structure of the economy is subject to recurrent shifts and knowledge is imperfect, agents must rely on alternative methods of anticipating future events and this may dramatically alter the policy trade-offs.

Least squares learning is a form of expectations formation that diverges to only a relatively modest degree from rational expectations and nests it as a limiting case; it therefore represents the natural alternative to full rationality. A recent stream of literature, epitomized by Orphanides and Williams (2002), has shown that imperfect knowledge makes stabilisation policies more difficult: strategies that would be efficient under rational expectations may end up performing miserably if agents with limited information have to learn adaptively about the economic environment. This happens when central banks put too much weight on output stabilisation, since overly activist policies are prone to generate episodes in which the public's inflation expectations become uncoupled from the policy objective. An additional complication arises from the fact that if knowledge is incomplete, committing to a systematic policy becomes problematic, because the private sector cannot easily verify whether the central bank is delivering on its promises, and the gains from commitment are severely reduced, because expectations, being backward rather than forward-looking, cannot be manipulated to increase policy effectiveness. Since commitment is no longer feasible, the issue then becomes what strategy the central bank should adopt to minimise the welfare losses associated with discretionary policy decisions, taking into account the expectations formation mechanism

---

<sup>1</sup> I would like to thank Margaret Bray and Fabio Buseti for insights and suggestions.

and the changing structure of the economy. In a recent paper, Athey, Atkeson and Kehoe (2003) investigate what is the optimal degree of monetary policy discretion under rational expectations. They show that when the monetary authority has private information on the state of the economy,<sup>2</sup> there is a tension between discretion and time inconsistency, since tight constraints on policy actions mitigate the time inconsistency problem but leave little room for fine-tuning. The authors show that the optimal policy has either bounded or no discretion: their main finding is that the optimal policy would take the form of an inflation cap limiting the range of admissible values of the rate of inflation.<sup>3</sup>

This paper focuses on how the relaxation of the rational expectations hypothesis and the assumption of recurrent shifts in the economic structure change the way monetary policy is set. It applies a principal-agent approach to the time-inconsistency problem: society (the principal) assigns a loss function to the central bank (the agent) that may differ from society's preferences, in order to improve the discretionary equilibrium. The underlying assumption is that it is possible to commit the monetary authority to a particular loss function, whereas minimisation of the loss function occurs under discretion. Society has standard quadratic preferences on output and inflation and can appoint a central banker endowed with either a quadratic or a lexicographic preference ordering, the latter choice being justified by the findings by Athey et al. (2003) that the optimal monetary policy under discretion takes the form of an inflation cap<sup>4</sup> and by the observation that the tenet of quadratic preferences does not fully capture either the mandate or the actual practice of central banks.

The main findings of the paper are the following. First, when agents do not possess complete knowledge on the structure of the economy and rely on an adaptive learning technology, the incentives and constraints facing the monetary authority change considerably

---

<sup>2</sup> The state can be interpreted, for instance, as a preference shock, which reduces the negative impact of a monetary stimulus on social welfare.

<sup>3</sup> Athey and her co-authors assume that there is a short-run trade-off between unemployment and inflation and that the monetary authority receives a private signal on the current state of the economy, which can be interpreted as private information of the policymaker regarding the impact on social welfare of a monetary stimulus. The higher the state, the larger the inflation surprise the central rate engenders to maximise social welfare. They demonstrate that under bounded discretion, for any state less than the cutoff state, the monetary authority chooses an inflation rate that increases with it, and for any state greater than this cutoff state, the monetary authority chooses some constant inflation rate regardless of its information. Under no discretion, the inflation rate is always set at the highest level.

<sup>4</sup> As shown in section 3, a policymaker whose preference ordering is lexicographic will tend to adopt strategies that constrain inflation to stay below an upper bound.

and a bias towards conservatism arises, suggesting that society is better off appointing a policymaker whose degree of inflation aversion is higher than its own: even with no intrinsic dynamics in the economy, agents' and the policymaker's attempts to learn adaptively introduce inertia into the system, which makes it costly for the central bank not to respond promptly and forcefully to shocks. Second, a policy that involves a cap on inflation is helpful in reducing output and inflation variability, but it is not uniformly superior to a strategy aimed at minimising a quadratic loss function: what matters for society's welfare is that the monetary authority's degree of inflation aversion is high enough to prevent expectations to fluctuate too much and for too long. Third, the bias against stabilisation policies and towards conservatism and the relative efficiency of alternative monetary strategies do not depend on whether the memory of the learning process is finite or infinite.

The paper is related to the literature in several ways. It builds on the finding by Athey et al. (2003) that the optimal constraints on discretion take the form of an inflation cap and on the analysis in Terlizzese (1999) showing that such a policy is implemented by a policymaker endowed with a lexicographic preference ordering. It parallels, under more general conditions, the works of Terlizzese (1999) and Driffil and Rotondi (2003) in deriving the properties of a monetary strategy that has price stability as its primary objective and does not allow inflation to exceed an upper limit. In addition, it models two-side learning along the lines of Evans and Honkapohja (2002) and uses the study of Orphanides and Williams (2002) as a benchmark for the quantitative analysis. The most closely related of these contributions is the work of Orphanides and Williams: model simulations are designed so as to replicate their experiments and the objective of this paper is largely the same as theirs, namely to understand how the economy responds to alternative monetary strategies when agents have bounded rationality and imperfect knowledge.

The original contribution of this work is to extend the findings of Orphanides and Williams. It seeks to do this in three different ways. First, it assumes that the policymaker as well as private agents has imperfect knowledge; under this framework, policy effectiveness ends up depending both on inflation and output variability so that the bias towards conservatism, if confirmed, cannot be attributed to the limited role of output volatility in the reference model. Second, it uses the theoretical insights on the optimal degree of monetary discretion to test whether society can increase welfare by appointing a policymaker

whose preference ordering is lexicographic rather quadratic. Moreover, since lexicographic preferences entail a bound on inflation, they can provide an effective tool for reducing fluctuations in inflation expectations under adaptive learning: indeed, according to Orphanides and Williams (2002), it is the uncoupling between expected and target inflation caused by activist policies that is responsible for the bias in favour of conservative ones. Third, it tests the assertion of the negative impact of imperfect knowledge on economic stabilisation under a set of alternative learning mechanisms, without relying exclusively on constant-gain algorithms, whose parameterisation should hinge on the nature and size of the shocks hitting the economy.

The paper is organised as follows. The following section discusses how incomplete knowledge and learning modify the choice set of the policymaker and examines on the drawbacks of assuming a quadratic loss function for modelling the monetary authority's preferences. Section 3 outlines the model used in the paper and contrasts the implications of assuming quadratic or, alternatively, lexicographic preferences. Section 4 introduces econometric learning and studies how different policies affect the speed at which learning algorithms converge to the rational expectations equilibrium. Section 5 presents some evidence, obtained by means of simulation, on the distortions on monetary policymaking caused by assuming that agents have bounded rationality; the focus is on whether adaptive learning induces a bias toward conservatism and on whether strategies featuring bounded discretion are actually welfare improving. Section 6 concludes.

## **2. Preferences and uncertainty**

The literature on monetary policy relies almost unanimously on the assumption that preferences of the monetary authorities are adequately described by a quadratic loss function. Quadratic preferences are justified not only on the grounds that they ensure analytical tractability but also because, for small deviations from the steady-state equilibrium, they provide a good approximation to more general utility functions. With quadratic preferences, large shocks are penalised proportionately more than small ones and certainty equivalence holds, so that the optimal policy is unaffected by additive uncertainty; moreover, quadratic utility makes imperfect observability of the state variables irrelevant in the choice of the loss-minimising control rule, as recently demonstrated by Svensson and Woodford (2000).

It is doubtful, however, that the characterisation of the policymaker's behaviour that follows from the assumption of a quadratic loss function adequately describes the way central banks operate. Al-Nowaihi and Stracca (2002) report official statements by central banks in which the importance of uncertainty about the state of the economy as a major element shaping policy decisions is apparent. They quote, for instance, the following claim from the Inflation Report of the Sveriges Riiksbank: "The element of uncertainty in the inflation assessment can accordingly influence monetary policy's construction. A high degree of uncertainty can be a reason for giving policy a more cautious turn." The statement is clearly at odds with the certainty equivalence principle and suggests that both additive and multiplicative uncertainty affect policy choices. A similar research perspective is adopted by Orphanides and Wilcox (1996), who try to infer the loss function of the Federal Reserve from its policy choices and communication strategy. They assert that official statements of the Federal Open Market Committee (FOMC) give some evidence that the Federal Reserve pursues the long-run objective of price stability adopting an *opportunistic approach to disinflation*: when inflation is moderate but still above the target value, the Fed tends not to take deliberate actions to reduce inflation, but rather to wait for external circumstances to deliver the desired additional deceleration of price dynamics. This approach to the conduct of monetary policy is not easily mapped into a loss function for the policymaker, but it clearly does not square with the most often used preference ordering. In particular, if the Phillips curve is linear and the central bank's loss function is quadratic in inflation and the output gap, the opportunistic approach is suboptimal. By means of an exercise in reverse engineering, Orphanides and Wilcox find that the loss function which is consistent with an opportunistic policy has two key attributes: path dependence and differential valuation of deviations from the inflation and output targets. The latter property causes the policymaker to concentrate on different policy objectives under different circumstances, specifically to focus on output volatility when inflation is low and price stabilisation when inflation is high.

Bray and Goodhart (2002) note that there is a logical inconsistency in assuming a quadratic loss function for the monetary authority. Since the policymaker is an agent appointed by the government, the worst penalty he faces, in case of failure, is being fired by his principal. Though dismissal is certainly painful, the pain is finite, so that when the likelihood of failure is high enough, the marginal disutility decreases with increasing deviations from the objective



agreed with the principal. To be plausible, the loss function attributed to the central bank should therefore be non-convex for extreme values of the target variables.

Terlizzese (1999) observes that objective evidence against the assumption of quadratic preferences can be found in the legal framework regulating the activity of some central bank. He considers the case of the European Central Bank (ECB). The Statute mandates the ECB to pursue multiple objectives, but ranks these according to their relative importance, as if the underlying preference ordering were of the lexicographic type: first comes price stability and then come the harmonious and balanced development of economic activities, sustainable and non-inflationary growth, a high level of employment and social protection, provided that these do not jeopardise the achievement of the primary objective. The main rationale for designing such a mandate was presumably to help the ECB inherit the anti-inflationary credibility of the Bundesbank.<sup>5</sup>

One noticeable characteristic of a lexicographic preference ordering, as shown in Terlizzese (1999), is that it translates into policies setting an upper limit on permissible inflation rates. Athey et al. (2003) show that such a form of bounded discretion exhibits optimal properties when the central bank cannot commit. They consider an economy whose social welfare function is quadratic and depends on the state of the economy. If the monetary authority has private information about the state, a tension arises between discretion and time-consistency: tight constraints on discretion mitigate the time-inconsistency problem, but leave little room for the central bank to fine-tune its actions to its private information. To maximise welfare, well-designed rules have to trade-off the two objectives, and the authors show that society can implement the optimal policy simply by legislating an inflation cap.

The role of preferences is magnified when agents have incomplete knowledge on how the economy works. Suppose some parameters in the model are time-varying: the central bank knows the distribution from which they are drawn but does not know their values. Accordingly, when the policymaker adjusts policy, he cannot be sure of the impact of his actions on the economy, since his moves affect the conditional variance as well as the conditional mean of inflation and output. As originally shown in Brainard (1967), this kind of uncertainty

---

<sup>5</sup> To support this view, Driffil and Rotondi (2003) quote the pre-EMU Statute of the Bundesbank assigning the monetary authority a hierarchy of objectives. According to the Statute, “safeguarding the currency” was the primary goal and “support to the general economic policy of the Federal Government, but only in so far as this is consistent with the aim of safeguarding the currency” was the secondary objective of the monetary authority.

can introduce some degree of caution in policy responses, the extent of which depends both on the sources of uncertainty and on the shape of the loss function. The issue gets even more complicated if one allows for some form of learning. Orphanides and Williams (2002) study the impact of least square learning on the optimal monetary policy in a simple AS-AD model. They show that, with imperfect knowledge, the ability of private agents to forecast inflation depends on the monetary policy in place, with forecast errors on average smaller when the central bank responds more aggressively to inflationary pressures; although expectations remain nearly efficient, learning raises the persistence of inflation and distorts the central bank's trade-off between inflation and output stabilisation. The paper by Orphanides and Williams assumes a stable environment and posits that agents use constant gain learning, which is especially appropriate for estimating time-varying coefficients. The implications for the conduct of monetary policy arising when the structure of the economy is subject to periodic shifts are studied by Ellison and Valla (2001), who focus on the impact of imperfect information on the degree of activism of the monetary authority. They show that, under rational expectations, uncertainty provides a motive for the policymaker to move more cautiously, but it also motivates an element of experimentation that favours activism. Sargent (1999), seeking a justification for the bouts of high inflation in the US in the late 1970s and early 1980s, considers the case of a central bank erroneously assuming that the inflation-unemployment trade-off is time-varying when in fact it is not. Assuming the policymaker learns adaptively,<sup>6</sup> he shows that adaptive learning may force an otherwise stable economy to cycle indefinitely between two equilibria, one characterised by stable prices, the other by high inflation.

It is apparent that by dropping the assumptions of perfect information and rational expectations, the efficiency of competing policies change substantially, with the relative merits depending on preferences, information structure and sources of uncertainty.

### **3. The model**

The model presented in this section has two basic ingredients: (i) the unobservability of the supply shock; (ii) an unknown and time-varying output-inflation trade-off. The model is first solved under rational expectations and then under adaptive learning.

---

<sup>6</sup> Sargent assumes perpetual adaptive learning, which means that recent data are given a greater weight.

### 3.1 *The structure of the economy*

The economy is characterised by an expectations-augmented Phillips curve relationship, linking inflation surprises  $\pi - \pi^e$  to (detrended) output  $y$ .

$$(1) \quad y = \alpha (\pi - \pi^e) + \varepsilon$$

Inflation is the policy instrument and is controlled without error by the monetary authority; the natural level of output is normalised at zero. Output also responds to a zero-mean supply shock  $\varepsilon$ , unobservable to the central bank and the private sector and uniformly distributed on the close interval  $[-\mu, \mu]$ .

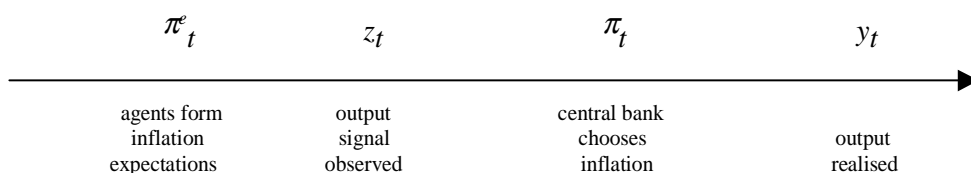
A Lucas-type aggregate supply function can be motivated as arising from the presence of one-period nominal wage contracts, set at the beginning of the period. One can derive (1) from the assumption that: (i) output is produced according to a Cobb-Douglas production function in which output depends on labour input; (ii) the nominal wage is set at the start of the period on the basis of inflation expectations at a level consistent with the labour market equilibrium; and (iii) labour demand is a function of ex-post real wages. Positive inflation surprises reduce labour costs and stimulate hiring and production, while negative surprises act in the opposite directions.<sup>7</sup>

The output shock  $\varepsilon$  is unobservable, but a signal  $z$ , conveying noisy information on  $\varepsilon$ , is observed by the policymaker after expectations have been determined, which ensures an information advantage to the central bank; it is assumed that  $z = \varepsilon + \xi$ , with  $\xi$  following a uniform distribution with the same support as  $\varepsilon$ , i.e.  $\xi \sim U[-\mu, \mu]$ .<sup>8</sup>

---

<sup>7</sup> One major drawback of the simple Lucas-type Phillips curve is that monetary policy actions have real effects only if they are unexpected. The current view, the so called New Neoclassical Synthesis, relies instead on the tenet that because inflation and inflation expectations are sticky, anticipated policy actions have real effects. Indeed, the more correctly anticipated the moves of the central bank, the greater their effectiveness. The simple Lucas-type Phillips curve, having no dynamics, is analytically tractable but, for this same reason, is unsuited to deal with stickiness of any sort.

<sup>8</sup> The assumption that both  $\varepsilon$  and  $\xi$  follow a uniform distribution ensures that in the rational expectations equilibrium a closed form solution for inflation expectations exists. The additional hypothesis that both shocks share the same support helps to keep the distribution of  $z$  simple. Indeed, the density function of  $z$  has a triangular shape with a kink point: had the two shocks been allowed to have equal means but different supports, the number of kinks would have increased and the density function of  $z$  would have become more complex (a trapezoid), making the computations of the equilibrium solution more cumbersome.



## 1. Timing of the model

The final ingredient of the model is the assumption that  $\alpha$ , the output-inflation trade-off, is a random variable. Since  $\alpha$  is time-varying, the effects of monetary policy on output depend on the value of the trade-off. It is assumed that  $\alpha = \bar{\alpha} + \tilde{\alpha} \sim IID(\bar{\alpha}, \sigma_\alpha^2)$  and that it is independent of all the other shocks in the economy.<sup>9</sup> Notice that the model is entirely static, so that no issue of strategic interaction between the monetary authority and the private sector arises.<sup>10</sup>

The timing of the model is shown in Figure 1. The signal  $z$  materialises before the central bank chooses the inflation rate but after private agents have set their inflation expectations for the period. The information advantage of the central bank creates a role for stabilisation policies and is meant to capture the fact that policy decisions can be made more frequently than are most wage and price decisions.

<sup>9</sup> The stochastic variable  $\alpha$  can be interpreted as an index of monetary policy effectiveness. It can be either discrete or continuous. What is relevant is the *IID* assumption, which avoids introducing dynamic elements into the optimisation problem of the central bank. Had  $\alpha$  been allowed to depend on its past realisations, the choice of the optimal policy would change, because of the strategic interactions arising between the private sector and the monetary authority. As shown in Ellison and Valla (2001), strategic interactions create a link between the activism of the central bank and the volatility of inflation expectations: the latter react to the former because an activist policy produces more information, helping private agents to learn. The value of experimentation in policymaking is also studied in Wieland (2003). Wieland, who assumes that agents do not have rational expectations but instead learn adaptively, stresses that two conflicting forces drive the optimal policy when model parameters are imperfectly known: on the one hand, uncertainty provides a motive for the policymaker to move cautiously; on the other hand, uncertainty also prompts an element of experimentation in policy. Wieland finds that the optimal policy that balances the cautionary and activist motives typically exhibits gradualism and is less aggressive than a policy disregarding parameter uncertainty. Strategic interactions among agents, though relevant in theory, may not be that important in practice and, in addition, make the analytic solution of the model impossible; they are therefore not considered in the paper.

<sup>10</sup> Balvers and Cosimano (1994) distinguish between passive and active learning policies, claiming that only the latter may be optimal. Active policies are those that incorporate the learning constraints in the loss-minimisation problem. They contend that the monetary policymaker should be more activist in its response to the observed state because this provides valuable information about the state of the economy: an activist central bank learns more quickly about the economy and is more effective in countervailing future output shocks. Active learning policies in the sense of Balvers and Cosimano are not considered in this paper, because they seem to be quite at odds with the limits imposed on agents' information set and processing capabilities. Moreover, the analytical framework is such that the distinction between active and passive learning policies is blurred: the model has no intrinsic dynamics and this wipes out any reward for policies that sacrifice current welfare for future gains.

### 3.2 Central bank loss function

The central bank is assumed to have either lexicographic or quadratic utility. In the principal-agent approach, society, whose preferences are quadratic, can assign a loss function to the central bank that may differ from its own: the final decision will depend on which option is best suited to improve the discretionary problem and maximise welfare.

Although in the general case a lexicographic preference ordering cannot be represented by a function, in the simplified case in which the monetary authority has been given only two objectives, such an ordering can be described by a loss function involving only the secondary objective, subject to a constraint involving the primary target. It is therefore assumed that the central bank aims at stabilising output around a non-zero level, provided that inflation is kept below a known upper bound.<sup>11</sup> Although there is some dispute on the correctness of formulations depicting central bankers as affected by an inflation bias, the assumption is retained because otherwise the only rational expectation for inflation would be the zero target itself and the inflation constraint would never be binding.

In formal terms, the problem solved by the central bank is

$$(2) \quad \begin{array}{l} \min_{\pi} \quad \frac{1}{2} E (y - k)^2 \\ s.t. \quad \begin{cases} \pi \leq \bar{\pi} \\ y = \alpha (\pi - \pi^e) + \varepsilon \end{cases} \end{array}$$

where  $k$  is the target level of output and the expectation operator is due to the unobservability of the output-inflation trade-off  $\alpha$  and the output shock  $\varepsilon$ . The assumption that  $k > 0$  is usually justified on the grounds that the presence of labour and goods market distortions leads to an inefficiently low level of output in equilibrium; alternatively,  $k > 0$  is interpreted as arising from political pressures on the central bank. Notice that  $k$  cannot exceed  $\mu$ , the upper bound of the output shock, which is the highest value achievable through inflation surprises. In what follows, it will be assumed that  $k$  is not too high and, in particular, that  $k = \frac{\mu}{6}$ . Under the standard hypothesis of time-separability of preferences, the problem is static and involves no trade-off between current and future utility, so that the optimal policy does not have to rely on

---

<sup>11</sup> The existence of a lower bound on inflation is neglected in this paper. In a model where the output shock is observable and the trade-off between output and inflation is time-invariant, Terlizzese (1999) shows that the main features of the monetary policy problem are largely unaffected by the inclusion of a lower bound on inflation. Intuitively, what explains this result is the asymmetric nature of the inflation bias that is assumed to characterise the monetary authority's preferences: if the central bank aims at pushing output above the natural level, it will tend to inflate, so that while the upper bound will often bite, the lower one will not.

the strategic interactions described by Ellison and Valla (2001), Bertocchi and Spagat (1993) and Wieland (2003).<sup>12</sup>

To highlight the implications of endowing the monetary authority with lexicographic preferences, the policy problem is also analysed under the standard assumption of quadratic loss function. In this case, the problem solved by the central bank can be formulated as

$$(3) \quad \begin{array}{l} \min_{\pi} \quad \frac{1}{2} E [(y - k)^2 + \beta \pi^2] \\ s.t. \quad y = \alpha (\pi - \pi^e) + \varepsilon \end{array}$$

where  $\beta$  measures the weight attached to the inflation objective relative to output stabilisation.

Regardless of the specific form of the loss function, be it (2) or (3), the model features an inflation bias, due to the policymaker's incentive to create surprise inflation so as to keep output above the natural level. However, economists think that since policymakers are not in the business of fooling people, such a feature makes the model irrelevant for monetary policy analysis.<sup>13</sup> Jensen (2003) tries to rescue Barro-Gordon type models by noting that their main implications, notably the inflation bias result, can be maintained without having to resort to the presumption that the central bank engenders inflation surprises to fool the public.<sup>14</sup>

### 3.3 *Signal extraction and the rational expectations equilibrium*

Given the structure of the problem, the issue of estimating the unobserved output shock and that of setting the optimal inflation rate can be kept separate and solved sequentially. Before deciding the optimal policy, the central bank has to solve a signal extraction problem. The first step is therefore to derive the probability distribution of  $z = \varepsilon + \xi$  and the conditional mean  $E(\varepsilon|z)$ . In Proposition 1 the density function of the signal  $z$  is derived, while in

---

<sup>12</sup> It is worth stressing that in Ellison and Valla (2001) the main advantage of active learning policies (i.e. policies that internalise the impact of central bank actions on private sector learning) is that they can facilitate the estimate of the unobserved output-inflation trade-off (the two-state Markov-switching process  $\beta_{s_t}$ ), which in turn affects the optimal response to the output signal. In addition, since regimes are persistent, a more accurate estimate of  $\beta_{s_t}$  helps in choosing the optimal inflation rate also in future periods. This is not the case for the model presented in this paper, since the output-inflation trade-off  $\alpha$  is assumed to be an *i.i.d.* process, which leaves no role to active learning policies.

<sup>13</sup> See, for instance, the quotation from Blinder, Vickers and Issing listed in Jensen (2003).

<sup>14</sup> The argument runs as follows. Assume that the monetary authority can control aggregate demand by means of the interest rate; then the Lucas-type supply equation can be inverted and transformed into an expectation-augmented Phillips curve. In equilibrium, the inflation bias comes along as a result of the attempt of the central bank to raise output (the unemployment rate) above (below) the natural rate and inflation surprises play no role.

Proposition 2 the first moment of the distribution of the output shock  $\varepsilon$  conditional on  $z$  is computed.

Proposition 1 *If  $z = \varepsilon + \xi$  and  $\varepsilon$  and  $\xi$  are independent uniform random variables, both defined on the interval  $[-\mu, \mu]$ , then the density function of  $z$  is equal to  $f(z) = \frac{1}{2\mu} + \frac{1}{4\mu^2} [\min(z, 0) - \max(0, z)]$ .*

**Proof.** See Appendix ■

Proposition 2 *If  $\varepsilon$  and  $\xi$  are uniform random variables, defined on the same close interval  $[-\mu, \mu]$ , and  $z = \varepsilon + \xi$ , then the optimal estimate of  $\varepsilon$  conditional on  $z$  is  $E(\varepsilon|z) = \frac{z}{2}$*

**Proof.** See Appendix ■

Given the assumption regulating the flow of information and the actions of the agents, the central bank sets the inflation rate on the basis of the observed signal and the private sector inflation expectations. Under lexicographic preferences, it will choose the inflation rate that solves the first-order condition  $E[\alpha(\alpha(\pi - \pi^e) + \varepsilon - k) | z] = 0$ , provided that the inflation constraint is satisfied, and will choose  $\pi = \bar{\pi}$  otherwise, i.e.

$$(4) \quad \pi = \begin{cases} \pi^e - \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_\alpha^2} \left( \frac{z}{2} - k \right) = \pi^e - \frac{z}{\phi} + \frac{2k}{\phi} & \text{if } z \geq 2k + \phi(\pi^e - \bar{\pi}) \\ \bar{\pi} & \text{otherwise} \end{cases}$$

where  $\phi^{-1} \equiv \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_\alpha^2} \frac{1}{2}$ . The optimal policy depends in a non-linear fashion on the value of  $z$ : when output shocks are strongly negative and the primary objective is at risk, the central bank acts as an inflation nutter; when the signal indicates more favourable disturbances, it behaves less conservatively and displays more activism, favouring output stabilisation. Notice that the optimal policy depends on the parameters of the distribution of the output-inflation trade-off  $\alpha$ . Two cases are considered, the first corresponding to the rational expectations equilibrium, the second introducing bounded rationality and least-square learning. First, it is assumed that  $\alpha$  is not observed but  $\bar{\alpha}$  and  $\sigma_\alpha^2$  are known by both the central bank and the private sector; this assumption could be justified if  $\varepsilon$ , though unobserved at the time expectations and the inflation rate are set, were observed with a one-period lag. Secondly, when learning is considered, the case when  $\bar{\alpha}$  and  $\sigma_\alpha^2$  (or, alternatively,  $\phi$ ) are to be estimated is also dealt with.

A few points illustrating the main properties of optimal policy are worth stressing. First, the optimal policy is not altered by the unobservability of the output shock, except

that the policymaker responds to an efficient estimate of the state vector rather than to its actual value. It is a well known result that a linear model with a quadratic loss function and a partially observable state of the economy is characterised by certainty-equivalence; since the assumed preference ordering is not quadratic, the result applies only when inflation is within the admissible range. Second, uncertainty about the multiplier of the policy instrument makes it optimal to react less than completely to the output shock. There is no gain in adopting more activist policies in order to learn from experimentation, since the model is static and the loss in current welfare incurred in overreacting is not compensated by smaller losses achievable in the future. The reduction in policy activism caused by parameter uncertainty, originally shown by Brainard (1967), reflects the direct impact of the monetary authority's instrument on the variability of the target variable.

For the equilibrium to be fully characterised, the solution for expected inflation must be provided. Under rational expectations, agents understand the incentives driving the actions of the central bank and set expectations that coincide on average with realisations. It therefore holds that  $\pi^e = \int \pi dF(z)$ , where  $F(z)$  is the distribution function of the signal  $z$ . Proposition 3 gives the full characterisation of the rational expectations equilibrium under the simplifying assumption that  $k = \frac{\mu}{6}$ .<sup>15</sup>

**Proposition 3** *If the central bank has lexicographic preferences and output is determined as in (1), there is a unique rational expectations equilibrium, where  $\pi = \min \left[ \pi^e - \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_{\alpha}^2} \left( \frac{z}{2} - k \right), \bar{\pi} \right]$  and  $\pi^e = \bar{\pi} - \frac{2k}{\phi} = \bar{\pi} - \frac{\mu}{3\phi}$ .*

**Proof.** See Appendix ■

Equilibrium is noncooperative Nash: the central bank and the private sector try to maximise their objective function taking as given the other player's actions. The assumption of rational expectations implicitly defines the loss function of the private sector as  $E(\pi - \pi^e)^2$ : given the public's understanding of the central bank's decision problem, its choice of  $\pi^e$  is the one minimising disutility.

---

<sup>15</sup> Setting  $k = \frac{\mu}{6}$  amounts to assuming that  $2k + \phi(\pi^e - \bar{\pi}) = 0$  and implies that the central bank chooses its best static response when  $z > 0$ , while it has no discretion for negative values of the signal. The value chosen for  $k$  is consistent with the common view that the inflation bias is not large; moreover, it makes it possible to solve for the rational expectations equilibrium analytically. Larger values, besides being unrealistic, would reduce the scope for stabilisation policies, but would not have a significant impact on the main findings of the paper.



From the expression for  $\pi^e$ , it is apparent that the existence of an upper bound on inflation contributes to stabilising expected inflation: for any value of  $k$ , the lower  $\bar{\pi}$ , the lower expected inflation. Notice that, unlike Rogoff's proposal of appointing a conservative central bank, delegating monetary policy to a policymaker endowed with lexicographic preferences does not impose a trade-off between reducing the inflation bias and stabilising the economy, provided that the output shock  $\varepsilon$  is not too negative. Another feature of the policy is that the larger the support of the output shock, the lower  $\pi^e$  and the closer it comes to zero. The intuition for this result is straightforward: positive (and higher than  $k$ ) output shocks trigger a reaction of the central bank, which creates negative inflation surprises to stabilise output, whereas large negative disturbances cannot be neutralised, because overly high inflation rates are not admissible. Widening the support of  $\varepsilon$  has an asymmetric effect on the actions of the monetary authority: it increases the cases in which the central banks finds it optimal to deflate, but has no influence on its incentives to inflate.

The equilibrium level of output is found by substituting the optimal policy and the solution for  $\pi^e$  in the supply function. In the simplifying case  $k = \frac{\mu}{6}$ ,

$$(5) \quad y = \begin{cases} \frac{2\alpha}{\phi}k + \varepsilon & = \frac{\mu}{3\phi}\alpha + \varepsilon & \text{if } z < 0 \\ \frac{2\alpha}{\phi}\left(k - \frac{z}{2}\right) + \varepsilon & = \frac{\mu}{3\phi}\alpha + \left(1 - \frac{\alpha}{\phi}\right)\varepsilon - \frac{\alpha}{\phi}\xi & \text{otherwise} \end{cases}$$

Two features of the optimal monetary policy are worth stressing. First, for values of the signal in the non-empty interval  $[-2\mu, 0)$ , inflation is constant and equal to  $\bar{\pi}$ , which is higher than  $\pi^e$ : the central bank keeps price dynamics above inflation expectations and in so doing it sustains output, though it cannot cushion against shocks. Second, even in the face of favourable output shocks, the policymaker is unable to stabilise output fully at the desired level. Two factors attenuate policy effectiveness: uncertainty about the output-inflation trade-off and unobservability of  $\varepsilon$ ; the former reduces the response of the central bank by a factor of  $\frac{2\alpha}{\phi}$ , while the latter leaves part of the output shock, namely  $\varepsilon - \frac{z}{2}$ , unchecked. Since  $\frac{z}{2}$  is an unbiased estimate of  $\varepsilon$ , unobservability of the output shock increases volatility but does not affect the degree of activism of the policy response; by contrast, unobservability of the output-inflation trade-off has a bearing on the policy strategy, since it favours more cautious policies, and is a source of permanent loss in welfare for the monetary authority.

Under (4), output is on average zero while its variance is equal to

$$(6) \quad E(y_{LEX}^2) = \left(1 - \frac{1}{4} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} - \frac{k}{\mu} \left(1 - 3 \frac{k}{\mu}\right) \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2}\right) \frac{\mu^2}{3} = \left(1 - \frac{1}{3} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2}\right) \frac{\mu^2}{3}$$

Output volatility is therefore smaller than  $\frac{\mu^2}{3}$ , the variance of the output shock, implying some degree of stabilisation on the side of monetary policy. To assess the distinguishing traits of the policy pursued by a central bank with lexicographic preferences, it is useful to contrast it with the optimal policy arising under the standard assumption of quadratic loss function. If (3) describes the central bank's problem, the policy instrument is set according to the rule

$$(7) \quad \pi = \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \pi^e + \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} (k - \frac{z}{2}) = \rho \pi^e + \frac{\rho}{\phi} (2k - z)$$

where  $\rho \equiv \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta}$ , with  $0 < \rho \leq 1$ . Taking the expected value of both sides of the equation, one finds that  $\pi^e = \frac{\bar{\alpha}}{\beta} k$ . Using the previous expression to substitute  $k$  out of the policy rule, it is easily seen that the optimal policy prescribes that the inflation rate is set according to the rule:

$$(8) \quad \pi = \pi^e - \frac{\rho}{\phi} z$$

The stochastic process for output, which is clearly mean zero, becomes  $y = -\frac{\alpha\rho}{\phi} z + \varepsilon = \left(1 - \frac{\alpha\rho}{\phi}\right) \varepsilon - \frac{\alpha\rho}{\phi} \xi$  and the corresponding variance is

$$(9) \quad E(y_{QUA}^2) = \left(1 + \frac{\bar{\alpha}^2}{2} \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{(\bar{\alpha}^2 + \sigma_\alpha^2 + \beta)^2} - \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta}\right) \frac{\mu^2}{3} = \left(1 - \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \left(1 - \frac{\rho}{2}\right)\right) \frac{\mu^2}{3}$$

A few differences are apparent when one compares the two optimal policies. In the presence of output shocks which are not too negative, rule (4) ensures more output stabilisation, as the increase in the inflation rate which must be engendered to counteract the supply disturbance does not have a negative impact on welfare and hence does not bring on a trade-off between the output and the inflation objectives. The reverse is true when  $\varepsilon$  is large and negative, because in that case output stabilisation is sacrificed to the primary objective of price stability. More activist policies are possible at the cost of larger inflation variability: as a general case, there exists a value  $\bar{\beta}$  such that, for  $\beta \in (\bar{\beta}, \infty)$ , output variability under lexicographic preferences (henceforth strategy 1) is lower than under quadratic ones

(henceforth strategy 2), and there exists a value  $\underline{\beta}$  such that, for  $\beta \in [0, \underline{\beta})$ ,  $E(\pi - \pi^e)^2$  is smaller under strategy 1. Since  $\underline{\beta} < \bar{\beta}$ , strategy 1 cannot outperform strategy 2 in terms of both objectives; however, a government whose loss function is quadratic can improve welfare by appointing a central bank endowed with lexicographic preferences and setting the upper bound  $\bar{\pi}$  so as to eliminate the inflation bias.

The next proposition states the conditions under which a central bank with lexicographic preferences is more effective in maximising social welfare than a policymaker endowed with the same quadratic loss function as society.

**Proposition 4** *Assume that a benevolent government, whose loss function is quadratic as in (3), appoints a central banker with lexicographic preferences as in (2). Compared with the case in which the monetary authority has the same preferences as society, the following results holds: (i) output volatility is lower for values of  $\beta$  larger than  $\frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\sqrt{3}-1}$ ; (ii)  $E(\pi - \pi^e)^2$  is smaller for values of  $\beta$  less than  $(\sqrt{3}-1)(\bar{\alpha}^2 + \sigma_\alpha^2)$ ; (iii) society's welfare is higher for values of  $\beta$  in a subset of the interval  $\left[0, \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\sqrt{3}-1}\right]$ .*

**Proof.** See Appendix ■

One property of the loss function (2) is worth stressing. Optimal policy under lexicographic preferences resembles closely inflation zone targeting:<sup>16</sup> under the latter, policy is characterised by a zone of inaction for small deviations of inflation from target. The width of the zone of inaction does not depend on the discount factor, but vanishes if the central banker is an inflation nutter. Zone targeting is derived by Orphanides and Wieland under either of two conditions: an ad-hoc loss function or a non-linear Phillips curve. Lexicographic preferences provide an alternative, less ad-hoc rationale for policies aiming at containing inflation within a target range rather than pursuing a point target.

#### 4. Adaptive learning and monetary policy regimes

In the real world, where shifts in policies and in the economic structure are by no means rare events, people often face the problem of understanding whether and how the environment

---

<sup>16</sup> Orphanides and Wieland (1999) call the current practice of several central banks to use target ranges instead of point targets “inflation zone targeting”. The distinctive feature of zone targeting is that when inflation is close to the bliss level, the policymaker may want to avoid deliberate attempts at changing aggregate demand that would be necessary for further improvements in inflation performance.

has changed and what is the least costly way to adapt decision rules to the new framework. In such a context, a strict application of the rational expectations hypothesis (REH) would not be a convincing theoretical solution. Alternatives have long been suggested. Herbert Simon (1957), for instance, supported some kind of bounded rationality and proposed to create a theory with behavioural foundations in which agents learned in the same way as econometricians. The basic difference between bounded rationality and the REH is that the latter restricts the set of available decision rules by assuming individual optimisation and consistency of beliefs, while the former replaces the hypothesis of consistency of perceptions with heuristic algorithms for representing and updating decision rules. However, doing without rational expectations yields to much freedom in modelling expectations and decision rules.

An increasingly important stream of literature, which builds on the pioneering work of Bray (1982)<sup>17</sup> and Marcet and Sargent (1989), recently revived in particular by Evans and Honkapohja,<sup>18</sup> has introduced a specific form of bounded rationality, called adaptive learning, where agents adjust their forecast rule as new data become available over time. This approach has several advantages, providing an asymptotic justification for the REH and allowing us to neglect non-learnable solutions in models with multiple equilibria. The availability of an alternative to rational expectations may be of great significance in practice, as for instance when the economy undergoes structural shifts and agents accordingly need to relearn the stochastic properties of the equilibrium solution; in such a case, dynamics does not disappear asymptotically and one can expect learning to remain important over time. In addition, although adaptive learning implies imperfectly rational expectations, as agents need to estimate the reduced form equations they employ, it can be viewed as a nearly rational expectations formation mechanism, in that forecasts are close to being efficient.

The key idea in adaptive learning is the assumption that at each period  $t$  private agents have a perceived law of motion (PLM) that they use to make forecasts. The PLM relates the variables of interest, whose future values are to be anticipated, to a set of state variables; the projection parameters are estimated using least squares. Forecasts generated in this way are

---

<sup>17</sup> Bray (1982) was the first to provide a result showing convergence to rational expectations in a model in which expectations influence the economy and agents use an econometric approach to update recursively their expectations.

<sup>18</sup> A large part of the contribution by Evans and Honkapohja to the literature on adaptive learning is summarised in Evans and Honkapohja (2001). Another useful reference is Sargent (1993).

used in decisions for period  $t$ , which yields the temporary equilibrium, also called the actual law of motion (ALM). The temporary equilibrium provides a new data point and agents are then assumed to re-estimate the projection parameters with data through period  $t$  and to use the updated forecast functions for period  $t+1$  decisions. The learning dynamics continues with the same steps in subsequent periods. Notice that adaptive learning to some extent accommodates the Lucas critique, in that expectations formation is endogenous and adjusts to changes in policy or structure.

The updating of the projection coefficients may be represented in terms of a system of recursive equations, having, under reasonable assumptions for the PLM, the rational expectations equilibrium as a fixed point. The recursive equations describe the mapping between the PLM and the ALM. Convergence of the adaptive learning process may be studied by means of the associated ordinary differential equation (ODE):<sup>19</sup> stability holds whenever the real parts of the eigenvalues of the Jacobian of the ODE are negative, i.e. whenever the system is E-stable.

The connection between E-stability and the convergence of least squares learning is a great advantage, since E-stability conditions are often easier to work out. However, focusing on asymptotic approximations puts aside any consideration of the learning speed. Issues concerning the speed of convergence of recursive least squares learning have been largely overlooked in the literature, mainly because there are just very few analytical results one can call forth. Marcet and Sargent (1992), building on the analysis by Benveniste, Metivier and Priouret (1990), relate the speed of convergence of the learning process to the eigenvalues of the associated ODE at the equilibrium point. The basic analytic result is the theorem stating that root- $t$  convergence<sup>20</sup> applies when the real part of the largest eigenvalue of the Jacobian of the ODE is less than  $-\frac{1}{2}$ . When this condition on the eigenvalues is not met, no analytic results on the asymptotic distribution are known, since the importance of initial conditions fails to vanish quickly enough.

---

<sup>19</sup> For large  $t$ , provided a few regularity conditions are met, the stochastic recursive algorithm is well approximated by an ordinary differential equation. These regularity conditions involve the stochastic process driving the state variables, the deterministic gain sequence and the function governing the revision in the projection coefficients.

<sup>20</sup> Root- $t$  convergence means convergence at a rate of the same order as the root square of the sample size.

In this section, adaptive learning is introduced to analyse the implications of imperfect knowledge on policy outcomes. The question to be answered is how the interaction between learning and central bank preferences affects aggregate welfare. Two cases are considered: in the first, only the private sector learns, while in the second both the private sector and the central bank have imperfect knowledge about the structure of the economy. To distinguish between the two cases, the first is called *single-agent learning* (SAL) and the second *two-agent learning* (TAL). The information advantage of the monetary authority is maintained: the policymaker moves last and sets the inflation rate after observing the signal  $z$  and private sector expectations. Private sector expectations are formed according to least squares learning and the monetary authority is assumed to behave fully rationally; in the TAL case, the central bank too has imperfect knowledge and uses recursive least squares to estimate the parameters it needs to set the inflation rate optimally.

#### 4.1 *Private sector learning*

Suppose that private agents have non-rational expectations, which they try to correct through adaptive learning. Assume also that the policymaker does not explicitly take agents' learning into account and continues to set policy according to either (4) or (7). The evolution of output and inflation is therefore described by the system

$$(10) \quad \begin{aligned} y &= \alpha \left( \pi - \widehat{E}^P \pi \right) + \varepsilon \\ \pi &= \begin{cases} \min \left[ \widehat{E}^{CB} \pi - \frac{z}{\phi} + \frac{2k}{\phi}, \bar{\pi} \right] \\ \rho \widehat{E}^{CB} \pi - \frac{\rho}{\phi} z + \frac{\rho}{\phi} 2k \end{cases} \end{aligned}$$

where the inflation rate depends on the monetary authority's preferences.  $\widehat{E}^P \pi$  represents the private sector's current estimate of the inflation rate, while  $\widehat{E}^{CB} \pi$  is the value of inflation expectations used in the central bank's control rule. It is assumed that private agents run regressions to set  $\widehat{E}^P \pi$ , while the monetary authority, which observes  $\widehat{E}^P \pi$  before moving, has rational expectations and therefore sets  $\widehat{E}^{CB} \pi = \widehat{E}^P \pi$ .

At each period  $t$ , private agents have a PLM for inflation that they use to make forecasts, which takes the form  $\widehat{E}^P \pi_t = a_{Pt}$ ,<sup>21</sup> where

$$(11) \quad a_{Pt} = a_{Pt-1} + \frac{1}{t} (\pi_{t-1} - a_{Pt-1})$$

The estimate  $a_{Pt}$  is updated over time using least squares;  $\frac{1}{t}$  represents the gain parameter, which is a decreasing function of the sample size.<sup>22</sup> Equation (11), in line with the literature, is in recursive form, uses data up to period  $t - 1$  and requires a starting value at time  $t = 0$ . The PLM has the same form as the RE solution for expected inflation: private agents estimate the parameter of the reduced form and set  $\widehat{E}^P \pi_t = a_{Pt}$ .

The issue now is to analyse the properties of the temporary equilibrium and its long-run behaviour. Consider first the case in which the central bank acts as if it had a lexicographic ordering of preferences. Given that the PLM is  $\widehat{E}^P \pi_t = a_{Pt}$ , the ALM turns out to be

$$\pi_t = \begin{cases} a_{Pt} - \frac{z_t}{\phi} + \frac{2k}{\phi} & \text{if } z_t \geq 0 \\ \bar{\pi} & \text{otherwise} \end{cases}$$

The mapping between the PLM and the ALM generates the stochastic recursive algorithm

$$a_{Pt} = \begin{cases} a_{Pt-1} + \frac{1}{t} \left( -\frac{z_{t-1}}{\phi} + \frac{2k}{\phi} \right) & \text{if } z_{t-1} \geq 0 \\ a_{Pt-1} + \frac{1}{t} (\bar{\pi} - a_{Pt-1}) & \text{otherwise} \end{cases}$$

which is approximated by the following ODE

$$\frac{d}{dt} a_P = h(a_P) = \lim_{t \rightarrow \infty} E(\pi_{t-1} - a_P)$$

where

$$\begin{aligned} \lim_{t \rightarrow \infty} E(\pi_{t-1} - a_P) &= (\bar{\pi} - a_P) \int_{-2\mu}^0 \left( \frac{1}{2\mu} + \frac{z}{4\mu^2} \right) dz + \int_0^{2\mu} \left( -\frac{z}{\phi} + \frac{2k}{\phi} \right) \left( \frac{1}{2\mu} - \frac{z}{4\mu^2} \right) dz \\ &= \frac{1}{2} (\bar{\pi} - a_P) + \frac{k - \mu/3}{\phi} \\ &= \frac{1}{2} (\bar{\pi} - a_P) - \frac{\mu}{6\phi} \end{aligned}$$

---

<sup>21</sup> A time index is used only when strictly necessary, as for instance when tracking the evolution over time of least squares learning.

<sup>22</sup> There are also algorithms featuring a constant gain parameter. Orphanides and Williams call these processes perpetual learning.

Notice that the fixed point of the ODE, namely  $a_P = h^{-1}(0) = \bar{\pi} - \frac{2k}{\phi} = \bar{\pi} - \frac{\mu}{3\phi}$ , coincides with the unique RE equilibrium for expected inflation. The theorems on the convergence of stochastic recursive algorithms can be applied so that convergence is governed by the stability of the associated ODE.<sup>23</sup> Since  $\frac{d}{da_P}h(a_P) = -\frac{1}{2} < 0$ , the ODE is (globally) stable and adaptive learning asymptotically converges to the RE equilibrium. Notice that the size of the eigenvalue of  $h(a_P)$  depends on the share of the support of  $z$  corresponding to an active policy: in general, root- $t$  convergence does not hold.

Consider now the case in which the central bank has quadratic preferences. The optimal policy for the monetary authority is to set  $\pi_t = \rho a_{Pt} - \frac{\rho}{\phi} z_t + \frac{\rho}{\phi} 2k$ . Compared to the RE case, the central bank does not wholly offset inflation expectations and the parameter  $k$  explicitly enters the control rule: both features disappear asymptotically, provided that  $a_P \rightarrow \pi^e = \frac{\bar{\alpha}}{\beta} k$ .

If agents use recursive least squares, then expectations evolve according to the equation

$$a_{Pt} = a_{Pt-1} + \frac{1}{t} (\pi_{t-1} - a_{Pt-1}) = a_{Pt-1} + \frac{1}{t} \left[ (\rho - 1) a_{Pt-1} - \frac{\rho}{\phi} z_{t-1} + \frac{\rho}{\phi} 2k \right]$$

and

$$h(a_P) = \lim_{t \rightarrow \infty} E \left[ (\rho - 1) a_{Pt} - \frac{\rho}{\phi} z_{t-1} + \frac{\rho}{\phi} 2k \right] = (\rho - 1) a_P + \frac{\rho}{\phi} 2k$$

Also in this case, the fixed point of the ODE, namely  $a_P = h^{-1}(0) = \frac{\rho}{(1-\rho)\phi} 2k = \frac{\bar{\alpha}}{\beta} k$ , coincides with the unique RE equilibrium and the system is (globally) stable. In fact,  $\frac{d}{da_P}h(a_P) = \rho - 1 < 0$ , since  $\rho$  is positive and smaller than one. Whether or not root- $t$  convergence holds depends on the size of  $\beta$ : the greater the weight the central bank attaches to the inflation objective, the faster agents learn.<sup>24</sup> The explanation of this result is quite intuitive: the attempt to offset output shocks requires generating inflation surprises, i.e. moving the inflation rate away from expectations, so that every period agents will have to revise their estimate with values of  $\pi$  that may substantially differ from the unconditional mean. A

---

<sup>23</sup> Chapter 6 of Evans and Honkapohja (2001) studies the conditions under which convergence of stochastic recursive algorithms is ensured. In the present case, they amount to show that the process  $\pi$ , which is a linear function of  $z$ , is bounded and stationary and that the function driving the updating of the projection parameter, namely  $\pi_{t-1} - a_{Pt-1}$ , is bounded and is twice continuously differentiable (with respect to both  $\pi_{t-1}$  and  $a_{Pt-1}$ ), with bounded second derivatives. Whether stability holds locally or globally depends on whether the regularity conditions hold on an open set around the equilibrium or for all admissible values of  $a_P$ . These regularity conditions, in the present case, are clearly met.

<sup>24</sup> A similar finding for the case of a New Keynesian model is obtained in (2004).



similar result applies to the case of lexicographic preferences: by reducing the support of the signal corresponding to an active policy (i.e. to a policy that seeks to avoid excessive output fluctuations), expectations adjust faster to the long-run equilibrium. Notice that for reasonable parameterisation of the model, the value of  $\beta$  which is required for  $\frac{d}{da_P}h(a_P)$  to be less than  $-\frac{1}{2}$  is large, meaning that only in the case of a highly inflation-averse central bank root- $t$  convergence holds.

The previous result is interestingly similar to the one in Orphanides and Williams (2002), who use a dynamic model based on an aggregate supply and an aggregate demand equation. They find that, with imperfect knowledge, the ability of private agents to forecast inflation depends on the monetary policy in place, with forecast errors smaller on average when the central bank responds more aggressively to inflationary pressures. Significantly improved economic performances can be achieved by placing greater emphasis on controlling inflation: more aggressive policies reduce the persistence of inflation and facilitate the formation of expectations, which in turn enhances economic stability and mitigates the influence of imperfect knowledge on the economy. The conclusion of the paper turns out to be quite similar to the Obstfeld solution to the central bank's credibility problem under discretion: to improve welfare, the responsibility of the conduct of monetary policy must be delegated to a policymaker who is more -averse than society.

#### 4.2 *Private sector and central bank learning*

Consider now the case where  $\bar{\alpha}$  and  $\sigma_\alpha^2$  (or, alternatively,  $\phi$ ) are not known to the policymaker. The central bank needs to estimate them since both parameters affect the policy rule through the degree of responsiveness to the signal  $z$ . As usual, it is assumed that they are gauged by means of least squares and that the estimate is updated every time new realisations of  $y$  and  $\pi$  are available. This form of bounded rationality corresponds to the case where  $\varepsilon$  is never observed, so that  $\bar{\alpha}$  and  $\sigma_\alpha^2$  cannot be directly estimated on the basis of past realisations of the output shock.

To account for central bank learning, the previous model must be augmented with a new set of recursive equations, which are the same irrespective of the monetary authority's preferences, as learning involves parameters rather than variables so that the values to be estimated are not related to agents' behaviour.

The system of recursive least squares equations is now the following

$$(12) \quad \begin{aligned} a_{Pt} &= a_{Pt-1} + \frac{1}{t} (\pi_{t-1} - a_{Pt-1}) \\ \widehat{\alpha}_t &= \widehat{\alpha}_{t-1} + \frac{1}{t} R_{\pi,t-1}^{-1} (\pi_{t-1} - a_{Pt-1}) \left[ \left( y_{t-1} - \frac{z_{t-1}}{2} \right) - \widehat{\alpha}_{t-1} (\pi_{t-1} - a_{Pt-1}) \right] \\ R_{y,t} &= R_{y,t-1} + \frac{1}{t} \left[ \left( y_{t-1} - \frac{z_{t-1}}{2} \right)^2 - R_{y,t-1} \right] \\ R_{\pi,t} &= R_{\pi,t-1} + \frac{1}{t} \left[ (\pi_{t-1} - a_{Pt-1})^2 - R_{\pi,t-1} \right] \end{aligned}$$

or, more compactly,

$$\theta_t = \theta_{t-1} + \frac{1}{t} Q (\theta_{t-1}, X_t)$$

where  $\theta_t = \left( a_{Pt}, \widehat{\alpha}_t, R_{y,t}, R_{\pi,t} \right)'$  and  $X_t = (1, \alpha_t, z_t, \varepsilon_t)'$ . The first equation is the same as in the previous section and captures private sector learning, while the others refer to the central bank's inference problem:  $\widehat{\alpha}_t$  is an estimate of the mean value of the output-inflation trade-off;  $R_{y,t}$  measures the sample variance of  $y - \frac{z}{2}$ , the policy-driven component of the output gap;<sup>25</sup>  $R_{\pi,t}$  is the second moment of the inflation surprise. As shown below, the central bank computes the statistics  $R_{y,t}$  and  $R_{\pi,t}$  as an intermediate step to estimate the optimal response coefficient to the signal  $z$  in the policy rule.

While the recursion for  $R_{\pi,t}$  is obvious, being simply the estimate of the variance of the inflation surprise, the other two equations require some explanation. To understand the recursion for  $\widehat{\alpha}_t$ , notice that the output equation can be rearranged as

$$y - \frac{z}{2} = \bar{\alpha} (\pi - a_P) + \left[ \varepsilon - \frac{z}{2} + \tilde{\alpha} (\pi - a_P) \right]$$

Since the central bank observes the signal  $z$ , it knows  $y - \frac{z}{2}$  and can efficiently estimate  $\bar{\alpha}$  by regressing it on the inflation surprise  $(\pi - a_P)$ . Using  $y - \frac{z}{2}$  as the regressand is effective because  $\varepsilon - \frac{z}{2}$  is orthogonal to the signal  $z$ , being the residual of the regression of  $\varepsilon$  on  $z$ , and to  $\pi - a_P$ , since the latter is a function of  $z$ . By assumption,  $\alpha$  is independent of any other stochastic variable in the model, so that the regression of the policy-driven component of the output gap,  $y - \frac{z}{2}$ , on  $\pi - a_P$  is a consistent and efficient (linear) estimator of  $\bar{\alpha}$ .

The justification for the recursion for  $R_{y,t}$  is somewhat more involved. A biased estimator of  $E(\alpha^2)$  can be obtained from the sample average of the squared (policy-driven

---

<sup>25</sup> Since  $E(\varepsilon|z) = \frac{z}{2}$ , the difference  $y - \frac{z}{2}$  represents (an unbiased estimate of) the share of the output gap which is not due to the supply shock  $\varepsilon$ , but depends on the inflation surprise.

component of the) output gap, scaled by the second moment of the inflation surprise

$$\frac{E\left(y - \frac{z}{2}\right)^2}{E\left(\pi - a_P\right)^2} = \frac{E\left(\alpha^2\right) E\left(\pi - a_P\right)^2 + E\left(\varepsilon - \frac{z}{2}\right)^2}{E\left(\pi - a_P\right)^2} = \bar{\alpha}^2 + \sigma_\alpha^2 + \frac{2\frac{\mu^2}{3}}{E\left(\pi - a_P\right)^2}$$

Since the bias depends on  $E\left(\pi - a_P\right)^2$  and on known parameters, it can be easily computed.  $R_{y,t}$  therefore represents an intermediate step in the computation of  $\psi_t \equiv \frac{R_{y,t} - 2\frac{\mu^2}{3}}{R_{\pi,t}}$ , the sample estimate of  $\bar{\alpha}^2 + \sigma_\alpha^2$ .<sup>26</sup>

Whether the stochastic recursive algorithm converges or not depends on the associated ODE, i.e. on the Jacobian of the matrix  $h(\theta) = \lim_{t \rightarrow \infty} EQ(\theta, X_t)$ . In the case of lexicographic preferences, we have

$$(13) \quad \begin{bmatrix} \frac{d}{d\tau} a_P \\ \frac{d}{d\tau} \hat{\alpha} \\ \frac{d}{d\tau} R_y \\ \frac{d}{d\tau} R_\pi \end{bmatrix} = h(\theta) = \begin{bmatrix} \frac{1}{2}(\bar{\pi} - a_P) - \frac{\hat{\alpha}}{2\frac{R_y - 2\mu^2/3}{R_\pi}} \frac{\mu}{6} \\ R_\pi^{-1} E(\pi - a_P)^2 \left(\bar{\alpha} - \hat{\alpha}\right) \\ E\left(y - \frac{z}{2}\right)^2 - R_y \\ E\left(\pi - a_P\right)^2 - R_\pi \end{bmatrix}$$

while for the standard quadratic case

$$(14) \quad \begin{bmatrix} \frac{d}{d\tau} a_P \\ \frac{d}{d\tau} \hat{\alpha} \\ \frac{d}{d\tau} R_y \\ \frac{d}{d\tau} R_\pi \end{bmatrix} = h(\theta) = \begin{bmatrix} -\frac{\beta}{\frac{R_y - 2\mu^2/3}{R_\pi} + \beta} a_P + \frac{\hat{\alpha}}{\frac{R_y - 2\mu^2/3}{R_\pi} + \beta} \frac{\mu}{6} \\ R_\pi^{-1} E(\pi - a_P)^2 \left(\bar{\alpha} - \hat{\alpha}\right) \\ E\left(y - \frac{z}{2}\right)^2 - R_y \\ E\left(\pi - a_P\right)^2 - R_\pi \end{bmatrix}$$

It is apparent that while the specific form of the loss function does not affect the inference problem of the central bank, it does have a bearing on private sector learning: agents use the inflation rate set by the monetary authority to update their inflation forecasts and are therefore influenced by the way the central bank behaves.

Both systems are recursive.  $R_\pi \rightarrow E(\pi - a_P)^2$  from any starting point, which implies that  $R_\pi^{-1} E(\pi - a_P)^2 \rightarrow I$ , provided that  $R_\pi$  is invertible along the path. The same happens for  $R_y$ . Hence, the stability of the differential equation for  $\hat{\alpha}$  may be assessed regardless of the remaining part of the system. Conditional on  $\hat{\alpha}$ ,  $R_y$  and  $R_\pi$  approaching the true

---

<sup>26</sup> One alternative could have been to regress  $\left(y - \frac{z}{2}\right)^2$  on  $\left(\pi - a_P\right)^2$ . The drawback of this approach is that the bias is a more convoluted function of the model parameters than in the case considered in the paper.

parameter values, convergence to the rational expectations equilibrium (REE) of private sector expectations is determined on the basis of the eigenvalues of the ODE for  $a_P$ . It is noticeable that the probability limit of the latter does not depend on the information set of the central bank and is the same whether or not the monetary authority knows the full structure of the economy. Conditions for learnability of the REE under both lexicographic and quadratic preferences are stated in the next proposition.

*Proposition 5* Assume that the economy is endowed with agents that rely on adaptive learning to form expectations; moreover, assume that the central bank has only incomplete information about the structure of the economy and uses recursive least squares (RLS) to estimate the unknown parameters. Then, the asymptotic behaviour of the system is described by (13) and (14) and, regardless of whether the policy maker has quadratic or lexicographic preferences, the discretionary rational expectations equilibrium is unique and E-stable: the estimates  $(\widehat{\alpha}_t, \psi_t)$  converge locally to  $(\bar{\alpha}, \bar{\alpha}^2 + \sigma_\alpha^2)$  and expectations of private agents tend in the limit to the RE values.

**Proof.** See Appendix. ■

As in the case when only the private sector learns, the effect of preferences on the speed of convergence is not clear. For low values of  $\beta$ , a central bank setting policy so as to minimise a quadratic loss function seems to be less effective in driving the economy towards the REE, while the opposite is true when  $\beta$  is high. In the TAL case, however, the central bank's imperfect knowledge introduces an additional layer of interaction between monetary policy and economic outcomes and model dynamics and properties cannot be properly analysed by focusing only on the asymptotic distribution of the estimated parameters. In particular, when the learning process is disturbed by several sources of shocks, only for large values of the sample size  $t$  the ODE becomes an acceptable approximation to the stochastic recursive algorithm and the asymptotic distribution is not of much help in understanding the properties of the system. The problem is even more serious in models where there are multiple equilibria, since in such cases, in early time periods, when estimates are based on very few degrees of freedom, large shocks can displace  $\theta_t$  outside the domain of attraction of the ODE and the system can therefore converge to any of the equilibrium points.<sup>27</sup> It follows that when

---

<sup>27</sup> When there is a unique equilibrium and the ODE is stable, it can be shown that  $\theta_t \rightarrow \theta^*$  with probability

agents' information set is severely constrained, as in the TAL case, both the asymptotic and the finite sample behaviour of the system are relevant. Theoretical results are therefore no longer sufficient and relying on simulation experiments and numerical results becomes unavoidable.

## 5. Imperfect knowledge and policy effectiveness

Model simulations are used to illustrate how learning affects the dynamic properties of inflation, inflation expectations and output. First the performance of the forecasting rules is assessed by focusing on the mean and the median of the inflation forecasts under different learning assumptions. Then the issue of the relative speed of convergence is considered; in so doing, the numerical procedure proposed by Marcet and Sargent (1992) is used. Finally, the output-inflation variability trade-off is computed and the role of policy regimes and learning is assessed. Each experiment is replicated for different values of the monetary authority's degree of inflation aversion. To account for the finding by Orphanides and Williams (2002), that policies that put too much weight on output stabilisation can generate episodes in which public's expectations of inflation become uncoupled from the policy objective, additional simulations are run mimicking the impact on the economy of a string of negative supply shocks; the interaction between the expectations formation mechanism and the degree of policy activism is studied under alternative assumptions about the central bank's preferences. Finally, as a further check on how much the results depend on the chosen learning mechanism, the assumption of infinite memory is dropped and the case of perpetual learning is considered.<sup>28</sup>

Each experiment is based on 500 replications; all simulations cover an interval of 2000 periods; subsamples of 500 observations are also considered in order to estimate the convergence speed. Initial conditions for the lagged variables in the RLS algorithm are randomly drawn from the distribution corresponding to the RE equilibrium. Results reported in the tables are computed on all but the first 150 observations of each replication, so as to minimise the impact of initial observations that can be too far away from the equilibrium

---

1 from any starting point. When there are multiple equilibria, however, such a strong result does not apply, unless one artificially constrains  $\theta_t$  to an appropriate neighborhood of the locally stable equilibrium  $\theta^*$ . In the earlier literature, local convergence was obtained by making an additional assumption on the algorithm, known as the *projection facility*. As a reference, see Evans and Honkapohja (2001), Section 6.4.

<sup>28</sup> Perpetual learning is sometimes used as a synonym of constant gain learning. A constant gain algorithm is preferable when agents believe that the economic environment is subject to frequent structural changes. In such a case, observations far away in the past are no longer informative and may even become a source of distortion.

solution. The model is calibrated according to the estimates in Ellison and Valla (2001); the selected parameter values are reported in Table 1.<sup>29</sup> Concerning  $\beta$ , the relative weight in the loss function of the inflation objective, three values are considered, namely  $\beta = \{.1765, 1, 5.666\}$ : the higher the value of  $\beta$ , the less willing is the central bank to pursue an activist policy and to tolerate large deviations of inflation from target and the lower is the inflation bias and hence  $\pi^e$ . Under lexicographic preferences, it is assumed that  $\bar{\pi}$  is chosen so as to drive inflation expectations to zero.

*Table 1:* Baseline calibrated parameters

Parameter	Value
$\bar{\alpha}$	1.75
$\sigma_{\alpha}$	0.5
$\mu$	0.0175
$k$	0.0029

Table 2a shows a few summary statistics on the agents' inflation forecasting model based on stochastic simulations, referring to the SAL case. In order to see how alternative monetary regimes affect the performance of least-squares inflation forecasts, the table reports - for lexicographic preferences and for the three selected values of  $\beta$  in the case of quadratic utility - the mean, the median and the standard deviation of (the distribution across realisations of) expected inflation; volatility measures are also reported for output and inflation. For ease of comparability, the equilibrium value of expected inflation under perfect information is shown in the first column. The table also contains estimates of the speed of convergence (to the REE) of the learning process.

Marcet and Sargent (1992) suggest a numerical procedure to obtain an estimate of the rate of convergence.<sup>30</sup> The starting point is the assumption that there is a  $\delta$  for which

$$(15) \quad t^{\delta} (\theta_t - \theta) \xrightarrow{D} F$$

---

<sup>29</sup> It is well known that, in its streamlined version, the Lucas supply function does not fit the data well. If one estimates the equation on United States data, even allowing for some richer dynamics than envisaged in (1), the correlation between the output gap and inflation surprises has the wrong sign. This evidence is quite robust and survives changes in the sample size, in the specification of the equation and in the proxies chosen for potential output and inflation expectations. In order to have an equation which is data-consistent, Ellison and Valla (2001) estimate a bivariate VAR model that proxies unobserved variables with deterministic components and allows for time-variability in the autoregressive parameters and in the covariance matrix.

<sup>30</sup> An interesting recent paper focusing on the speed of adjustment of adaptive learning algorithms is Ferrero (2004).

where  $\theta_t$  is the vector of parameters of the PLM,  $\theta$  is its asymptotic limit and  $F$  is some non-degenerate, well-defined zero-mean distribution.  $\delta$  is the rate of convergence of  $\theta_t$  since, for any  $\bar{\delta} < \delta$ ,  $t^{\bar{\delta}} (\theta_t - \theta) \rightarrow 0$ . Letting  $\sigma_F^2$  denote the variance under the distribution  $F$ , (15) implies that  $E [t^\delta (\theta_t - \theta)]^2 \rightarrow \sigma_F^2$  and

$$\frac{E [t^\delta (\theta_t - \theta)]^2}{E [(tl)^\delta (\theta_{tl} - \theta)]^2} \rightarrow 1$$

which, in turn, implies that

$$\frac{E (\theta_t - \theta)^2}{E (\theta_{tl} - \theta)^2} \rightarrow l^{2\delta}$$

It follows that, for large  $t$ , it is justified to consider

$$\delta = \frac{1}{\log l} \log \sqrt{\frac{E (\theta_t - \theta)^2}{E (\theta_{tl} - \theta)^2}}$$

as an approximation to the rate of convergence.

Given  $t$  and  $l$ , the expectations can be approximated by simulating a large number of independent realisations of length  $t$  and  $tl$ , and calculating the mean square across realisations. The approximated value of the convergence rate of the learning algorithm is also reported in the table.

As shown in the table, for most values of the preference parameter  $\beta$ , a central bank acting so as to minimise a lexicographic loss function seems to be more effective in driving private sector expectations towards the perfect knowledge equilibrium value: a policymaker optimising a quadratic loss function cannot succeed in anchoring expectations so tightly. However, for high values of  $\beta$  expected inflation under least-squares learning becomes nearly identical to the value implied by the perfect knowledge benchmark and the variability of inflation expectations is reduced sizeably. The greater precision of the forecasts under strategy 1 is also confirmed by the smaller standard deviation of  $a_P$ . One possible interpretation of this evidence is that the very existence of an upper bound on inflation helps to stabilise agents' expectations. Notice also that the estimates of  $a_P$  are slightly biased upward under strategy 2. A further result worth mentioning is that, regardless of the monetary policy regime, the mean and the median of expected inflation coincide, showing that the distribution across realisations of the forecasts is not skewed.

Regarding the speed of convergence of the learning mechanism  $\delta$ , no strategy is uniformly superior: for intermediate values of  $\beta$ , a policy based on lexicographic preferences seems to be more effective than one based on quadratic utility in driving private sector expectations towards the REE, while for extreme values of  $\beta$  the opposite result holds. It is worth noticing, however, that convergence is nearly immediate, which suggests that the value of  $\delta$  might somewhat underestimate the actual effectiveness of the learning process and the differences across estimates might be fairly spurious.

In most cases, strategy 1 seems to be welfare improving, though for high values of the inflation aversion parameter  $\beta$  the two strategies becomes nearly indistinguishable. The welfare effects of departing from the benchmark of perfect knowledge are negligible: the standard deviation of both output and inflation is basically the same as under rational expectations.

To summarise the main findings of the experiment, one can conclude that when the economic environment is not too complex and the learning process is not hampered by too many sources of noise, deviations from the rational expectations assumption do not cause substantial welfare losses. Convergence to the equilibrium is quick and the volatility introduced by imperfect knowledge is negligible. For most values of the preference parameter  $\beta$ , it turns out that a benevolent government maximising society's quadratic utility function does better to appoint a central banker whose preferences are lexicographic. It pays to adopt a policy framework that puts a cap on actual (and expected) inflation and that eliminates the inflation bias.

The picture does not change substantially when the central bank too has imperfect knowledge. Tables 3a and 4a report the simulation results for the two-agent learning model: the former table describes the "plain" RLS learning rule (UE), while the latter shows results for the case of constrained estimation (CE).<sup>31</sup>

---

<sup>31</sup> Convergence of the learning process to the REE holds almost surely when there is a unique solution and the ODE is globally stable; in the more general case, convergence with probability 1 is guaranteed only when a "projection facility" is used, i.e. when  $\theta_t$  is artificially constrained to remain in an appropriate neighbourhood of  $\theta$ . The hypothesis of a projection facility however is inappropriate for decentralised markets and some alternative, less arbitrary device is to be preferred. The alternative adopted in the paper is dubbed "constrained estimation" and boils down to impose some minimal restriction on the admissible regions of the estimates of  $\hat{\alpha}_t$  and  $\psi_t \equiv \frac{R_{y,t} - 2\frac{\mu^2}{3}}{R_{\pi,t}}$ : in the first case it is assumed that only positive numbers are admissible, since surprises cannot have a negative impact on output; in the second, that only values greater than  $\frac{\hat{\alpha}_t^2}{\alpha_t^2}$  are sensible, since variances cannot be



The standard deviation of output and inflation, expected inflation and the estimate of the policy rule coefficient are reported in the tables and so is the convergence speed for  $a_P$ ,  $\widehat{\alpha}_t$  and  $\psi_t$ . For ease of comparison, the corresponding values for the REE are also shown. One of the most striking findings of the simulation experiment is the minor effect on output and inflation variability of assuming imperfect central bank knowledge: compared with the case in which only the private sector learns, the increase in output and inflation volatility is surprisingly small, usually just a handful of percentage points. This is remarkable, since the estimation problem faced by the monetary authority is quite convoluted, requiring dealing with non-linearities and computing higher-order moments. In the vast majority of cases, the increase in volatility of output and inflation remains well below 10% regardless of the preferences of the central bank, suggesting that the cost for the policymaker of having partial knowledge of the working of the economy is not disproportionately large. It is worth recalling however that the model lacks intrinsic dynamics, which explains why deviations from the REE tend to be short-lived.

Concerning the relative efficiency of the two strategies, in neither case does dropping the assumption of perfect knowledge strongly affect the performance of the optimal rule: welfare remains quite close to the REE outcome. For low values of  $\beta$ , a central bank endowed with lexicographic preferences is more effective in keeping inflation expectations under control and it is also successful in stabilising output fluctuations, even when bounds are not imposed on the RLS algorithm. The situation is reversed when  $\beta$  is high. A downward bias is evident in the estimate of the parameter measuring the response to the signal  $z$ , but it mostly disappears in the CE case; the imprecision in guessing the value of  $\frac{1}{\phi}$  is responsible for some undesired fluctuations in output, while the excessive volatility of inflation is not attributable to the surprise component but rather to movements in private-sector inflation expectations, which under adaptive learning are not constant as in the REE. Since the bounds imposed on the RLS algorithm mimic the working of a projection facility, the rejection rate in the CE case turns out to be substantially lower.<sup>32</sup> Indeed, because of the complexity of the filtering problem

---

negative. Both assumptions represent minimal rationality requirements and though they cannot guarantee almost sure convergence of the learning algorithm, they can in principle contribute to reduce the number of non-convergent replications.

<sup>32</sup> The rejection rate is computed on the basis of the RLS estimate of the second moment of the output-inflation trade-off. Replications are considered as diverging if the estimate of  $\bar{\alpha}^2 + \sigma_\alpha^2$  is at least three times greater than the true value. The initial 150 observations are not used in the computation. In the case of lexicographic

facing the monetary authority, in a large number of replications shocks displace the recursive algorithm outside the domain of attraction of the ODE and the estimate of the optimal response coefficient in the policy rule remains far from the true value. If, for any reason, at time  $t$  the estimate of  $\phi$  is very large, the monetary authority has no incentive to respond aggressively to the signal  $z$  and changes in  $y$  mostly reflect output shocks  $\varepsilon$ : in such occurrences, the data become uninformative about the output-inflation trade-off and the estimate of  $\phi$  remains the same indefinitely. Expectations become self-fulfilling and the economy gets stuck indefinitely on a suboptimal path, characterised by too passive a monetary policy, as if the policymaker's degree of inflation aversion were enormously higher than society's.

Except for high  $\beta$ s, strategy 2 underperforms strategy 1. It is, however, more effective in enhancing agents' learning process, as shown by the more precise estimate of the coefficient of the policy rule, which guarantees that the equilibrium under imperfect knowledge very closely matches the REE. There is no clear evidence that overly low or high inflation aversion can negatively impact on the accuracy of the estimate of  $\frac{\rho}{\phi}$ . Concerning the speed of convergence, the two rules are more or less equivalent, though under CE strategy 1 seems to be preferable;  $\delta$  is very close to one half, so that convergence to a Gaussian distribution of both sequences  $\{\theta_t\}^{LEX}$  and  $\{\theta_t\}^{QUA}$  cannot be ruled out. As expected, strategy 2 reduces output variability more than strategy 1 when  $\beta$  is low, while for high values the opposite result holds.

Table 2b, 3b and 4b present evidence for the case of perpetual learning; the statistics for the speed of convergence are not shown of course, since under constant-gain learning  $\theta_t$  may at most converge to a probability distribution but not to a non-stochastic point. No meaningful differences are apparent with respect to the previous case. Given the structure of the model, there is no advantage to discarding observations, so it is no surprise that in most cases RLS estimates are less accurate and stabilisation policies less effective.

An additional set of simulations have been run to analyse the dynamic response of output and inflation to a sequence of unanticipated shock. The experiment is designed by subjecting

---

preferences, absent constraints, the rejection rate turns out to be quite high (some 20%); it falls by a factor of 4 if the RLS algorithm is augmented with lower bounds. In the case of quadratic preferences, the number of diverging replications is on average much smaller and so is the gain obtained by imposing constraints on the learning process; for high  $\beta$ s, however, the rejection rate rises and becomes similar to the one observed under lexicographic preferences. The estimated number of diverging replications decreases substantially if less restrictive criteria are used. Notice that divergence pertains to central bank learning and is defined in terms of the estimates of the policy parameters  $\frac{1}{\phi}$  and  $\frac{\rho}{\phi}$ , which become very close to zero: neither the output gap nor the inflation rate are actually deviating boundlessly from equilibrium.

the economy to a string a negative output shock that gradually decline in magnitude and vanish after 12 periods. With rational expectations, the impact of the shocks is short-lived and causes only a temporary fall in output and a rise in inflation, while under imperfect knowledge the response of the economy is prolonged and amplified by agents' learning. The objective of the experiment is to test whether the evidence reported by Orphanides and Williams (2002), namely that activist policies end up causing perceived inflation to become uncoupled from the policymaker's objectives, is valid and also holds for the theoretical framework adopted in this paper.

Table 5a reports the outcome of the experiment. Regardless of the central bank preferences, activist policies do not seem to pay off: the lower is  $\beta$ , the more volatile are inflation and output. Simulation evidence supports Rogoff's claim that appointing a central banker who places greater relative weight on the inflation objective than society does is welfare-improving. The result holds for both the UE and CE case. The evidence also confirms the finding that even if society is not too strongly inflation averse, it ought to appoint a central banker who strongly dislikes inflation, the specific form of preferences (i.e. lexicographic or quadratic) being of second-order importance. The similarities with the findings of Orphanides and Williams are surprising, considering the differences in the theoretical framework. First, the model adopted in this paper has no intrinsic dynamics and the only source of persistence comes from the assumption that agents learn adaptively: the uncoupling between actual and perceived inflation is much harder to achieve with so simple a dynamic structure, although the lack of dynamics in the economy is presumably compensated by the inertia induced by the attempts of the central bank to estimate the mean and variance of the output-inflation trade-off. Second, though only inflation expectations have a direct impact on the equilibrium outcome, output gap uncertainty affects the central bank's estimates of the moments of  $\alpha$  and hence the policy setting: it is by no means obvious that a strategy that penalises output variability might be conducive to greater welfare. The explanation of the existence of a bias in favour of hawkishness is to be found in the role of central bank learning: overly activist policies reduce the information content of the output gap and make estimates of the coefficients of the policy rule too volatile and unreliable. Unlike Orphanides and Williams, no effects are traceable to the size of the gain parameter: as shown in Table 5b, the same results are established under either infinite memory adaptive learning or perpetual leaning.

## 6. Conclusions

When the economy is subject to recurrent structural shifts, the monetary authority cannot credibly commit to a systematic approach to policy, since consistency between promises and actions is not easily verifiable, and agents cannot have rational expectations, since their knowledge of the surrounding environment is incomplete. Moreover, if the central bank has access to private information on the state of the economy, it will attempt to fine-tune its policy, and the information advantage of the monetary authority may end up creating a tension between discretion and time inconsistency. This paper has focused on the implications for the effectiveness of monetary policymaking of departing from the benchmark of rational expectations and has applied a principal-agent approach to deal with the time-inconsistency problem that arises when the central bank cannot commit. Special attention has been paid to validating two claims, namely that (i) policies designed to be efficient under rational expectations can perform very poorly when knowledge is incomplete and agents learn adaptively, and (ii) the optimal degree of monetary policy discretion is obtained with policies that put a cap on inflation.

The main results presented in the paper are the following. First, when agents do not possess complete knowledge on the structure of the economy and rely on an adaptive learning technology, the incentives and constraints facing the monetary authority change considerably and a bias toward conservatism arises, suggesting that society is better off appointing a policymaker whose degree of inflation aversion is higher than its own. The rationale for this finding is that, even in models with no intrinsic dynamics, agents' and the policymaker's attempts to learn adaptively introduce inertia into the system and can induce prolonged deviations of output and inflation from target, thus raising the costs for the central bank of not responding promptly and forcefully to shocks. Under both rational expectations and perfect knowledge, the more persistent is inflation, the more responsive to inflationary pressures are optimal policies: since adaptive learning introduces an additional source of inertia, it is a straightforward consequence that optimal policies under incomplete knowledge are more inflation averse than they are under full information. A bias against activist policies is also a feature of sticky-information models: Branch et al. (2005) show that at a sufficiently low level of activism, the trade-off between price and output volatility may disappear; accordingly, a reduction in price volatility can make it unnecessary for agents to update information too

frequently, leading in turn to a reduction in output variance. The findings of this paper therefore seem consistent with the stream of literature that stresses the direct relationship between incomplete information and the degree of inflation aversion in optimal policies. Second, a policy that involves a cap on inflation is helpful in reducing output and inflation variability, but it is not uniformly superior to a strategy aimed at minimising a quadratic loss function if the degree of inflation aversion is high enough. Third, the bias against stabilisation policies and towards conservatism, and the relative efficiency of alternative monetary strategies, do not depend on whether agents have finite or infinite memory: under both perpetual and decreasing-gain learning, incomplete knowledge considerably increases the costs of activist policies and the benefits of a tight control of inflation.

## Appendix

**Proof of Proposition 1.** Consider first two random variables,  $u$  and  $v$ , defined on the unit segment  $[0, 1]$ . Their sum,  $w = u + v$ , is defined on the close interval  $[0, 2]$ . The distribution function of  $w$ , for  $0 \leq w \leq 1$ , is given by  $H(w) = \int_0^w du \int_0^{w-u} dv = \frac{w^2}{2}$ , while, for value of  $w$  comprised in the interval  $(1, 2]$ , it is equal to  $H(w) = \int_0^w du \int_0^{w-u} dv - \int_1^w du \int_0^{w-u} dv - \int_1^w dv \int_0^{w-v} du = 1 - \frac{(2-w)^2}{2}$ . The corresponding density function is  $h(w) = w$  for  $0 \leq w \leq 1$  and  $h(w) = 2 - w$  for  $1 < w \leq 2$ , or, more compactly,  $h(w) = \min[w, 1] - \max[0, w - 1]$ .

Consider now the case in which the two random variables, rather than having support on the unit interval, are both defined on  $[-\mu, \mu]$ . One can write  $\varepsilon = -\mu + 2\mu x$  and  $\xi = -\mu + 2\mu y$ ; their sum,  $z = \varepsilon + \xi = -2\mu + 2\mu(x + y) = -2\mu + 2\mu w$ , is a linear transformation of the random variable  $w$ , i.e.  $z = g(w)$ . If  $f(z)$  denotes the density function of the variable  $z$ , the change-of-variable technique can be used to compute the density function of the variable  $z$ , i.e.  $f(z) = \left| \frac{d}{dz} g^{-1}(z) \right| h(g^{-1}(z))$ . Since  $g^{-1}(z) = 1 + \frac{z}{2\mu}$  and  $\left| \frac{d}{dz} g^{-1}(z) \right| = \frac{1}{2\mu}$ , one has that

$$\begin{aligned} \text{for } -2\mu \leq z \leq 0 & \quad f(z) = \left(1 + \frac{z}{2\mu}\right) \left| \frac{d}{dz} \left(1 + \frac{z}{2\mu}\right) \right| = \frac{1}{2\mu} + \frac{z}{4\mu^2} \\ \text{for } 0 \leq z \leq 2\mu & \quad f(z) = \left[2 - \left(1 + \frac{z}{2\mu}\right)\right] \left| \frac{d}{dz} \left(1 + \frac{z}{2\mu}\right) \right| = \frac{1}{2\mu} - \frac{z}{4\mu^2} \end{aligned}$$

which can be written in a more compact way as  $f(z) = \frac{1}{2\mu} + \frac{1}{4\mu^2} [\min(z, 0) - \max(0, z)]$ . The corresponding distribution function is

$$\begin{aligned} \text{for } -2\mu \leq z \leq 0 & \quad F(z) = \frac{1}{8\mu^2} (2\mu + z)^2 \\ \text{for } 0 \leq z \leq 2\mu & \quad F(z) = 1 - \frac{1}{8\mu^2} (2\mu - z)^2 \end{aligned}$$

**Proof of Proposition 2.** By definition,  $E(\varepsilon|z) = \int_D \varepsilon f(\varepsilon|z) d\varepsilon = \int_D \varepsilon \frac{f_1(z|\varepsilon)f_2(\varepsilon)}{f_3(z)} d\varepsilon$ , where  $f_i(\cdot)$ ,  $i = 1, 2, 3$ , denotes a (conditional or marginal) density function, and  $D$  is the domain of  $\varepsilon|z$ , which clearly depends on the current realisation of the signal  $z$ . The density function of  $z$  conditional on  $\varepsilon$  is the same as the density function of  $\xi$ , which is  $\frac{1}{2\mu}$ , while the probability law of  $z$ , as shown in Proposition 1, is  $f_3(z) = \frac{1}{2\mu} + \frac{1}{4\mu^2} [\min(z, 0) - \max(0, z)]$ .

It follows that

$$E(\varepsilon|z) = \int_D \varepsilon \frac{\frac{1}{2\mu} \cdot \frac{1}{2\mu}}{\frac{1}{2\mu} + \frac{1}{4\mu^2} [\min(z, 0) - \max(0, z)]} d\varepsilon = \int_D \frac{\varepsilon}{2\mu + \min(z, 0) - \max(0, z)} d\varepsilon$$

The remaining problem is to find  $D$ , the support of  $\varepsilon|z$ . Since  $\varepsilon = z - \xi$ , a given value  $\bar{z}$  of the signal shifts the support of the random variable  $\varepsilon$ , which becomes  $D = [-\mu, \mu] \cap [-\mu + \bar{z}, \mu + \bar{z}]$ . Two cases are possible, depending on whether  $\bar{z}$  is positive or negative. If  $z = -\bar{z}_1 < 0$ , then  $\varepsilon \in [-\mu, -\bar{z}_1 + \mu]$  and

$$E(\varepsilon|z) = \int_{-\mu}^{-\bar{z}_1 + \mu} \frac{\varepsilon}{2\mu - \bar{z}_1} d\varepsilon = \frac{1}{2\mu - \bar{z}_1} \frac{\varepsilon^2}{2} \Big|_{-\mu}^{-\bar{z}_1 + \mu} = \frac{1}{2\mu - \bar{z}_1} \frac{(-\bar{z}_1 + \mu)^2 - \mu^2}{2} = \frac{1}{2\mu - \bar{z}_1} \frac{\bar{z}_1(\bar{z}_1 - 2\mu)}{2} = -\frac{\bar{z}_1}{2} = \frac{z}{2}$$

If, instead,  $z = \bar{z}_2 > 0$ , then  $\varepsilon \in [-\mu + \bar{z}_2, \mu]$  and

$$E(\varepsilon|z) = \int_{-\mu + \bar{z}_2}^{\mu} \frac{\varepsilon}{2\mu - \bar{z}_2} d\varepsilon = \frac{1}{2\mu - \bar{z}_2} \frac{\varepsilon^2}{2} \Big|_{-\mu + \bar{z}_2}^{\mu} = \frac{1}{2\mu - \bar{z}_2} \frac{\mu^2 - (\bar{z}_2 - \mu)^2}{2} = \frac{1}{2\mu - \bar{z}_2} \frac{\bar{z}_2(2\mu - \bar{z}_2)}{2} = \frac{\bar{z}_2}{2} = \frac{z}{2}$$

Notice that, although the distribution of the variables is not normal, the optimal estimate for the unobserved shock  $\varepsilon$  is the same as in the standard case of Gaussian variables.

**Proof of Proposition 3.** If the output shock is not too unfavourable, the central bank's problem has an internal solution, obtained from the first-order condition  $E[\alpha(\alpha(\pi - \pi^e) + \varepsilon - k)] = E(\alpha^2)(\pi - \pi^e) + E(\alpha)\left(\frac{z}{2} - k\right) = 0$ . The optimal inflation rate is therefore

$$\pi = \pi^e - \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_\alpha^2} \left(\frac{z}{2} - k\right) = \pi^e - \frac{1}{\phi} (z - 2k)$$

where  $\phi^{-1} \equiv \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_\alpha^2} \frac{1}{2}$ . If instead the signal  $z$  indicates a value of  $\varepsilon$  close to  $-\mu$ , the value of  $\pi$  minimising the loss function is not admissible and the central bank will choose  $\pi = \bar{\pi}$ . The optimal strategy for the monetary authority is therefore the one described by (4).

The upper bound on inflation implies that  $\pi = \pi^e - \frac{z}{\phi} + \frac{2k}{\phi} \leq \bar{\pi}$ , which holds for  $z \in [2k + \phi(\pi^e - \bar{\pi}), 2\mu]$ : the higher  $k$ , the higher expected inflation and the lower the probability that the central bank succeeds in stabilising output. Since the density function of the signal has a kink at zero, it matters for the computation of expected inflation whether  $2k + \phi(\pi^e - \bar{\pi})$  is positive or negative, for its value determines how the support of the signal is split.

In the general case, to determine  $\pi^e$  it is necessary to solve a third-order polynomial, but the analysis is greatly simplified when  $2k + \phi(\pi^e - \bar{\pi})$  is equal to zero. Under this assumption, expected inflation is the solution to the following equation

$$\begin{aligned}\pi^e &= \int_{-2\mu}^0 \bar{\pi} \left( \frac{1}{2\mu} + \frac{z}{4\mu^2} \right) dz + \int_0^{2\mu} \left( \pi^e - \frac{z-2k}{\phi} \right) \left( \frac{1}{2\mu} - \frac{z}{4\mu^2} \right) dz \\ &= \frac{\bar{\pi}}{2} + \frac{1}{2} \left( \pi^e + \frac{2k}{\phi} \right) - \int_0^{2\mu} \frac{z}{\phi} \left( \frac{1}{2\mu} - \frac{z}{4\mu^2} \right) dz \\ &= \frac{\bar{\pi}}{2} + \frac{1}{2} \left( \pi^e + \frac{2k}{\phi} \right) - \frac{\mu}{3\phi}\end{aligned}$$

In equilibrium, expected inflation is therefore  $\pi^e = \bar{\pi} + \frac{2k}{\phi} - \frac{2\mu}{3\phi}$ , which simplifies to

$$\pi^e = \bar{\pi} - \frac{\mu}{3\phi}$$

once one uses the restriction  $2k + \phi(\pi^e - \bar{\pi}) = 0$  to substitute  $k$  out from the previous expression. Notice also that the implied value of the target level of output is  $k = \frac{\phi(\bar{\pi} - \pi^e)}{2} = \frac{\mu}{6}$ .

**Proof of proposition 4.** Under lexicographic preferences, the output level corresponding to the optimal policy is (5). Exploiting the fact that  $\alpha$  is independent of  $\varepsilon$  and  $\xi$  and that  $\varepsilon$  enters (5) additively, it follows that

$$\begin{aligned}E(y_{LEX}) &= \int_{-2\mu}^0 \frac{2\bar{\alpha}}{\phi} k \left( \frac{1}{2\mu} + \frac{z}{4\mu^2} \right) dz + \int_0^{2\mu} \left( \frac{2\bar{\alpha}}{\phi} k - \frac{\bar{\alpha}}{\phi} z \right) \left( \frac{1}{2\mu} - \frac{z}{4\mu^2} \right) dz \\ &= \frac{2\bar{\alpha}}{\phi} k \int_{-2\mu}^0 \left( \frac{1}{2\mu} + \frac{z}{4\mu^2} [\min(z, 0) - \max(0, z)] \right) dz - \int_0^{2\mu} \frac{\bar{\alpha}}{\phi} z \left( \frac{1}{2\mu} - \frac{z}{4\mu^2} \right) dz \\ &= \frac{2\bar{\alpha}}{\phi} k - \frac{\bar{\alpha}}{\phi} \frac{\mu}{3} \\ &= 0\end{aligned}$$

where the last equality uses the fact that, by assumption,  $k = \frac{\mu}{6}$ . Since the private sector anticipates the attempts of the policymaker to raise output above the natural level and adjusts expectations accordingly, output is on average zero. To compute the variance of  $y$ , it is helpful to split the random variable  $z$  in terms of the noise and signal components. This means that when any function  $h(z)$  of the variable  $z$  appears, the integral  $\int_{-2\mu}^0 h(z) \left( \frac{1}{2\mu} + \frac{z}{4\mu^2} \right) dz$  is substituted with the double integral  $\int_{-\mu}^{\mu} \int_{-\mu}^{-\varepsilon} h(\varepsilon + \xi) \frac{d\varepsilon}{2\mu} \frac{d\xi}{2\mu}$  and the



integral  $\int_0^{2\mu} h(z) \left( \frac{1}{2\mu} - \frac{z}{4\mu^2} \right) dz$  becomes  $\int_{-\mu}^{\mu} \int_{-\mu}^{\mu} h(\varepsilon + \xi) \frac{d\varepsilon d\xi}{2\mu 2\mu}$ . It follows that

$$\begin{aligned}
E(y_{LEX}^2) &= \frac{1}{4\mu^2} \left[ \int_{-\mu}^{\mu} \int_{-\mu}^{-\varepsilon} \int_{-\infty}^{\infty} \left( \frac{2\alpha}{\phi} k + \varepsilon \right)^2 d\Phi(\alpha) d\varepsilon d\xi + \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left( \frac{2\alpha}{\phi} k + \varepsilon - \frac{\alpha}{\phi} (\varepsilon + \xi) \right)^2 d\Phi(\alpha) d\varepsilon d\xi \right] \\
&= \frac{1}{4\mu^2} \left[ \int_{-\mu}^{\mu} \int_{-\mu}^{-\varepsilon} \int_{-\infty}^{\infty} \left( \frac{2\alpha}{\phi} k + \varepsilon \right)^2 d\Phi(\alpha) d\varepsilon d\xi \right. \\
&\quad \left. + \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left[ \left( \frac{2\alpha}{\phi} k + \varepsilon \right)^2 + \left( \frac{\alpha}{\phi} (\varepsilon + \xi) \right)^2 - 2 \left( \frac{2\alpha}{\phi} k + \varepsilon \right) \left( \frac{\alpha}{\phi} (\varepsilon + \xi) \right) \right] d\Phi(\alpha) d\varepsilon d\xi \right] \\
&= \frac{1}{4\mu^2} \left[ \underbrace{\int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left( \frac{2\alpha}{\phi} k + \varepsilon \right)^2 d\Phi(\alpha) d\varepsilon d\xi}_{A_1} + \underbrace{\int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left[ \frac{\alpha}{\phi} (\varepsilon + \xi) \right]^2 d\Phi(\alpha) d\varepsilon d\xi}_{A_2} \right. \\
&\quad \left. - 2 \underbrace{\int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left( \frac{2\alpha}{\phi} k + \varepsilon \right) \frac{\alpha}{\phi} (\varepsilon + \xi) d\Phi(\alpha) d\varepsilon d\xi}_{A_3} \right] \\
&= \frac{1}{4\mu^2} [A_1 + A_2 - 2A_3]
\end{aligned}$$

where  $\Phi(\alpha)$  is the distribution function of  $\alpha$ . The three integrals turn out to be equal to

$$\begin{aligned}
A_1 &= \left[ 16 (\bar{\alpha}^2 + \sigma_\alpha^2) \frac{k^2}{\phi^2} \mu^2 + \frac{4}{3} \mu^4 \right] \\
A_2 &= \left[ \frac{4}{3} \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\phi^2} \mu^4 \right] \\
A_3 &= \left[ \frac{8}{3} \frac{k}{\phi^2} (\bar{\alpha}^2 + \sigma_\alpha^2) \mu^3 + \frac{2}{3} \frac{\bar{\alpha}}{\phi} \mu^4 \right]
\end{aligned}$$

which implies that

$$\begin{aligned}
E(y_{LEX}^2) &= \frac{1}{4\mu^2} [A_1 + A_2 - 2A_3] \\
&= \left( 1 - \frac{1}{4} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} - \frac{k}{\mu} \left( 1 - 3 \frac{k}{\mu} \right) \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} \right) \frac{\mu^2}{3} = \left( 1 - \frac{1}{3} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} \right) \frac{\mu^2}{3}
\end{aligned}$$

where again the assumption  $k = \frac{\mu}{6}$  has been used. It is easily seen that  $E(y^2) < \frac{\mu^2}{3} = Var(\varepsilon)$ , showing that monetary policy succeeds in offsetting part of the volatility induced by the output shock.

If instead the central bank's loss function is quadratic, the first-order condition for optimality becomes  $E(\alpha^2)(\pi - \pi^e) - E(\alpha)(k - E(\varepsilon)) + \beta\pi = 0$  and the policy rule

maximising welfare is

$$\pi = \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \pi^e + \frac{\bar{\alpha}}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \left( k - \frac{z}{2} \right)$$

The private sector knows the incentives shaping the monetary authority's choices and sets expected inflation so as not to be fooled on average. It follows that  $\pi^e = \frac{\bar{\alpha}}{\beta} k$  and  $\pi = \pi^e - \frac{\rho}{\phi} z$ , where  $\rho \equiv \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta}$ . Substituting the expressions for inflation and expected inflation into the supply curve, it follows that the RE solution for output is

$$y = -\alpha \frac{\rho}{\phi} z + \varepsilon = \left( 1 - \alpha \frac{\rho}{\phi} \right) \varepsilon - \alpha \frac{\rho}{\phi} \xi$$

which is zero on average.

Under quadratic preferences, output volatility is equal to

$$\begin{aligned} E(y_{QUA}^2) &= \frac{1}{4\mu^2} \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left[ \left( 1 - \alpha \frac{\rho}{\phi} \right) \varepsilon - \alpha \frac{\rho}{\phi} \xi \right]^2 d\Phi(\alpha) d\varepsilon d\xi \\ &= \frac{1}{4\mu^2} \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \int_{-\infty}^{\infty} \left[ \left( 1 + \frac{\rho^2}{\phi^2} \alpha^2 - 2\frac{\rho}{\phi} \alpha \right) \varepsilon^2 + \frac{\rho^2}{\phi^2} \alpha^2 \xi^2 - 2\frac{\rho}{\phi} \left( \alpha - \frac{\rho}{\phi} \alpha^2 \right) \varepsilon \xi \right] d\Phi(\alpha) d\varepsilon d\xi \\ &= \frac{1}{4\mu^2} \int_{-\mu}^{\mu} \int_{-\mu}^{\mu} \left[ \left( 1 + \frac{\rho^2}{\phi^2} E\alpha^2 - 2\frac{\rho}{\phi} \bar{\alpha} \right) \varepsilon^2 + \frac{\rho^2}{\phi^2} E\alpha^2 \xi^2 - 2\frac{\rho}{\phi} \left( \bar{\alpha} - \frac{\rho}{\phi} E\alpha^2 \right) \varepsilon \xi \right] d\varepsilon d\xi \\ &= \frac{1}{4\mu^2} \int_{-\mu}^{\mu} \left[ \left( 1 + \frac{\rho^2}{\phi^2} (\bar{\alpha}^2 + \sigma_\alpha^2) - 2\frac{\rho}{\phi} \bar{\alpha} \right) 2\mu \varepsilon^2 + \frac{\rho^2}{\phi^2} (\bar{\alpha}^2 + \sigma_\alpha^2) \frac{2}{3} \mu^3 \right] d\varepsilon \\ &= \frac{1}{4\mu^2} \left[ \left( 1 + \frac{\rho^2}{\phi^2} (\bar{\alpha}^2 + \sigma_\alpha^2) - 2\frac{\rho}{\phi} \bar{\alpha} \right) \frac{4}{3} \mu^4 + \frac{\rho^2}{\phi^2} (\bar{\alpha}^2 + \sigma_\alpha^2) \frac{4}{3} \mu^4 \right] \\ &= \left( 1 + 2\frac{\rho^2}{\phi^2} (\bar{\alpha}^2 + \sigma_\alpha^2) - 2\frac{\rho}{\phi} \bar{\alpha} \right) \frac{\mu^2}{3} \\ &= \left( 1 - \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \left( 1 - \frac{1}{2} \rho \right) \right) \frac{\mu^2}{3} \end{aligned}$$

Comparing the output variance under the two alternative policy frameworks, one can assess under which conditions a central bank endowed with lexicographic preferences is more successful in stabilising output than a policymaker whose loss function is a standard quadratic one.

$$\begin{aligned} E(y_{QUA}^2) - E(y_{LEX}^2) &= \left( \frac{1}{3} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} - \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \left( 1 - \frac{1}{2} \rho \right) \right) \frac{\mu^2}{3} \\ &= \left( \frac{1}{3} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} - \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} \rho \left( 1 - \frac{1}{2} \rho \right) \right) \frac{\mu^2}{3} \\ &= \frac{1}{2} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} \left( \rho^2 - 2\rho + \frac{2}{3} \right) \frac{\mu^2}{3} \end{aligned}$$

The relative magnitude of the output variances depends on the value of  $\rho$ , which in turn depends on  $\beta$ . The second-order polynomial  $\rho^2 - 2\rho + \frac{2}{3}$  has two positive zeros, namely

$1 \pm \sqrt{\frac{1}{3}}$ . Since  $0 < \rho \leq 1$ , only the smallest root is admissible and, by simple manipulations, one finds that for values of  $\beta$  in the interval  $\in \left[ \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\sqrt{3}-1}, \infty \right)$ ,  $E(y_{QUA}^2) \geq E(y_{LEX}^2)$ , while for  $\beta \in \left[ 0, \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\sqrt{3}-1} \right)$  the inequality is reversed.

Concerning the variance of inflation, it is easily seen that the two variances are

$$\begin{aligned} E(\pi - \pi^e)_{LEX}^2 &= \frac{2}{3\phi^2} \left( \frac{\mu^2}{3} \right) \\ E(\pi - \pi^e)_{QUA}^2 &= 2\frac{\rho^2}{\phi^2} \left( \frac{\mu^2}{3} \right) \end{aligned}$$

It follows that  $E(\pi - \pi^e)_{QUA}^2 - E(\pi - \pi^e)_{LEX}^2 = \frac{2}{\phi^2} \left( \frac{\mu^2}{3} \right) (\rho^2 - \frac{1}{3}) > 0$  if and only if  $\beta \in \left[ 0, (\sqrt{3}-1)(\bar{\alpha}^2 + \sigma_\alpha^2) \right)$ . Since  $(\sqrt{3}-1)(\bar{\alpha}^2 + \sigma_\alpha^2) < \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\sqrt{3}-1}$ , it is therefore plain that strategy 1 cannot simultaneously reduce the variance of both output and inflation. What is relevant for social well-being is however  $E\pi^2 \equiv E(\pi - \pi^e)^2 + (\pi^e)^2$  and not  $E(\pi - \pi^e)^2$  and by properly setting the value of the upper bound  $\bar{\pi}$  a central banker endowed with lexicographic preferences can reduce the term  $(\pi^e)^2$  to zero. Assuming that society has quadratic preferences over output and inflation variability, as in (3), strategy 1 is more effective in promoting welfare if

$$\begin{aligned} \Delta W(\beta) &= E(y_{QUA}^2) - E(y_{LEX}^2) + \beta \left( E(\pi)_{QUA}^2 - E(\pi^2)_{LEX} \right) \\ &= \left( \frac{1}{2} \frac{\bar{\alpha}^2}{\bar{\alpha}^2 + \sigma_\alpha^2} (\rho^2 - 2\rho + \frac{2}{3}) + \beta \left[ \frac{1}{12} \left( \frac{\bar{\alpha}}{\beta} \right)^2 + \frac{2}{\phi^2} (\rho^2 - \frac{1}{3}) \right] \right) \frac{\mu^2}{3} \end{aligned}$$

Since  $\lim_{\beta \rightarrow 0} \Delta W(\beta) = +\infty$  and  $\Delta W \left( \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\sqrt{3}-1} \right) < 0$ , there are cases in which a government endowed with a quadratic loss function can improve social welfare by delegating the conduct of monetary policy to a central banker acting so as to minimise a lexicographic loss function.

**Proof of proposition 5.** Regardless of the preferences of the monetary authority, the recursive system representing the learning process is of the form  $\theta_t = \theta_{t-1} + \frac{1}{t} Q(\theta_{t-1}, X_t)$ , where  $\theta_t = (a_{Pt}, \hat{\alpha}_t, R_{y,t}, R_{\pi,t})'$  and  $X_t = (1, \alpha_t, z_t, \varepsilon_t)$ . To show the asymptotic stability of the REE under learning, one has to proceed as follows: first, it must be verified that there exists a non-trivial open domain containing the equilibrium point where the learning algorithm satisfies a few regularity conditions concerning the updating function  $Q(\theta_{t-1}, X_t)$  and the stochastic process driving the state variables  $X_t(\theta_{t-1})$ ; second, the local (or global) stability of the ODE associated with the stochastic recursive system must be established.

Consider first the case of lexicographic preferences. System (12) has a unique equilibrium point  $\theta^*$ , at which  $a_P = \bar{\pi} - \frac{\mu/3}{\phi}$ ,  $\hat{\alpha} = \bar{\alpha}$ ,  $R_\pi = \frac{2}{3\phi^2} \left( \frac{\mu^2}{3} \right)$  and  $R_y = (\bar{\alpha}^2 + \sigma_\alpha^2) \left[ \frac{2}{3\phi^2} \left( \frac{\mu^2}{3} \right) \right] + 2\frac{\mu^2}{3}$ . It can be easily seen that  $\theta^*$  is the REE. The stochastic process  $X_t(\theta_{t-1})$  is white noise, with finite absolute moments, so that regularity conditions (B.1) and (B.2) in Evans and Honkapohja (2001) are satisfied.<sup>33</sup> In addition, the gain sequence approaches zero asymptotically and is not summable. Finally, provided that  $R_\pi$  and  $R_y$  are non zero along the learning path,  $Q(\theta_{t-1}, X_t)$  satisfies a Lipschitz condition<sup>34</sup> on a compact set containing the equilibrium point  $\theta^*$ , which ensures that regularity conditions (A.1)-(A.3) in Evans and Honkapohja (2001) are also met. Convergence of the learning process to the REE therefore hinges on the stability of the associated ODE (13). Notice that the system is recursive and the asymptotic behaviour of the subsystem describing central bank learning can be assessed independently of the expectations formation mechanism of the private agents. Indeed, provided that  $R_\pi$  and  $R_y$  are invertible along the convergence path,  $R_y \rightarrow E(y - \frac{z}{2})^2$  and  $R_\pi \rightarrow E(\pi - a_P)^2$  from any starting point; since  $R_\pi^{-1} E(\pi - a_P)^2 \rightarrow I$ , it is easily seen that  $\hat{\alpha}_t \rightarrow \bar{\alpha}$ , since the eigenvalue of the Jacobian of the corresponding differential equation has a negative real part. Conditional on  $\hat{\alpha}_t \rightarrow \bar{\alpha}$ , convergence of private sector inflation forecasts follows, since the associated ODE is stable.

A more formal proof of the convergence of the learning process to the REE requires proving that the Jacobian of the ODE, evaluated at the REE  $\theta^*$ , has eigenvalues whose real part is negative. In order to show that this is indeed the case, first notice that  $R_\pi = E(\pi - a_P)^2$  at  $\theta^*$ , that implies that  $R_\pi$  does not appear in the first three equations of the ODE evaluated at  $\theta^*$ . A similar result holds for  $R_y$ . In the last two equations, the derivatives of  $R_\pi$  and  $R_y$  (and, accordingly, of  $E(\pi - a_P)^2$  and  $E(y - \frac{z}{2})^2$ ), though different from zero, cancel out, so that the Jacobian has the following upper triangular, block-recursive structure:

$$Dh(\theta^*) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{\bar{\alpha}^2 + \sigma_\alpha^2} \frac{\mu}{12} & \frac{3}{2\mu} \frac{1}{\bar{\alpha}} & -\frac{3}{2\mu} \frac{\bar{\alpha}^2 + \sigma_\alpha^2}{\bar{\alpha}} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

<sup>33</sup> Chapter 6 in Evans and Honkapohja (2001) lists the regularity conditions required for the analysis of the asymptotic behaviour of the stochastic recursive algorithm. Local stability is treated in section 6.2, global convergence in section 6.7.

<sup>34</sup>  $Q(\theta_{t-1}, X_t)$  satisfies a Lipschitz condition if it is bounded and twice continuously differentiable, with bounded second derivatives.

It is easily checked that its eigenvalues are  $(-\frac{1}{2}, -1, -1, -1)$ . They are all negative and the system is therefore E-stable.

Consider now the case of quadratic preferences. The unique equilibrium point  $\theta^*$  of the system (12) is now  $a_P = \frac{\bar{\alpha}}{\beta}k$ ,  $\widehat{\alpha} = \bar{\alpha}$ ,  $R_\pi = E(\pi - a_P)^2 = 2\left(\frac{\rho}{\phi}\right)^2\left(\frac{\mu^2}{3}\right)$  and  $R_y = E\left(y - \frac{z}{2}\right)^2 = (\bar{\alpha}^2 + \sigma_\alpha^2) \left[2\left(\frac{\rho}{\phi}\right)^2\left(\frac{\mu^2}{3}\right)\right] + 2\frac{\mu^2}{3}$ . The stochastic process  $X_t(\theta_{t-1})$  is independent of  $\theta$  and is the same as in the previous case, so that regularity conditions (B.1) and (B.2) in Evans and Honkapohja (2001) are satisfied. The same holds for the assumptions (A.1)-(A.3) on the gain sequence and the updating function  $Q(\theta_{t-1}, X_t)$ . The stability of the associated ODE (14) can be proved in the same way as for the system (13): provided that  $R_\pi$  and  $R_y$  are invertible along the convergence path,  $R_\pi \rightarrow E(\pi - a_P)^2$  from any starting point; central bank estimates converge to the true parameter values  $\bar{\alpha}$ , since the eigenvalue of the Jacobian of the corresponding differential equation has a negative real parts, and  $a_P \rightarrow \frac{\bar{\alpha}}{\beta}k$ , since the associated ODE is stable.

As in the previous case, the structure of the Jacobian justifies the sequential solution of the system. At  $\theta^*$ , the derivative matrix of the ODE (14) is equal to

$$Dh(\theta^*) = \begin{bmatrix} -\frac{\beta}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} & \frac{1}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta} \frac{\mu}{6} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The lower block for  $R_\pi$  and  $R_y$  can be solved first; then, triangularity of the upper block ensures that convergence for  $\widehat{\alpha}_t$  does not depend on the asymptotic behaviour of  $a_{Pt}$ . The eigenvalues of the Jacobian are  $\left(-\frac{\beta}{\bar{\alpha}^2 + \sigma_\alpha^2 + \beta}, -1, -1, -1\right)$  and the system is therefore E-stable.

## References

- Al-Nowaihi A. and L. Stracca (2002), “Non-standard central bank loss functions, skewed risks and certainty equivalence”, European Central Bank, *Working paper no. 129*.
- Athey, S. , A. Atkeson and P.J. Kehoe (2003), “The Optimal Degree of Monetary Policy Discretion”, Federal Reserve Bank of Minneapolis, *Research Department Staff Report*.
- Balvers, R.J. and T.F. Cosimano (1994), “Inflation variability and gradualist monetary policy”, *Review of Economic Studies*.
- Benveniste, A., M. Metivier and P. Priouret (1990), *Adaptive Algorithms and Stochastic Approximations*, Springer-Verlag, Berlin Heidelberg.
- Bertocchi, G. and M. Spagat (1993), “Learning, experimentation and monetary policy”, *Journal of Monetary Economics*..
- Brainard, W. (1967), “Uncertainty and the Effectiveness of Policy”, *American Economic Review*.
- Branch, W.A., J. Carlson, G.W. Evans and B. McGough (2005), “Monetary Policy, Endogenous Inattention and the Volatility Trade-off”, mimeo.
- Bray, M.M. (1982), “Learning, Estimation and Stability of Rational Expectations”, *Journal of Economic Theory*.
- Bray, M. and C. Goodhart (2002), “You Might as Well be Hung for a Sheep as a Lamb: The Loss Function of an Agent”, *FMG Discussion Paper no. 418*.
- Driffil, J. and Z. Rotondi (2004), “Monetary Policy and Lexicographic Preference Ordering”, *Centre for Economic Policy Research, Discussion Paper no. 4247*.
- Ellison, M. and N. Valla (2001), “Learning, uncertainty and central bank activism in an economy with strategic interactions”, *Journal of Monetary Economics*.
- Evans, G.W. and S. Honkapohja (2001), *Learning and Expectations in Macroeconomics*, Princeton University Press.
- Evans, G.W. and S. Honkapohja (2002), “Expectations and the Stability Problem for Optimal Monetary Policy”, mimeo.
- Ferrero, G. (2004), “Monetary policy and the transition to rational expectations”, Banca d’Italia, *Temi di discussione no. 499*.
- Jensen, H. (2003), “Explaining an Inflation Bias without Using the Word *Surprise*”, mimeo.
- Marcet, A. and T.J. Sargent (1989), “Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models”, *Journal of Economic Theory*.

- Marcet, A. and T.J. Sargent (1992), "Speed of convergence of Recursive Least Squares Learning with ARMA Perceptions", *UPF Economics Working Paper no. 15*.
- Orphanides, A. and D.W. Wilcox (1996), "The Opportunistic Approach to Disinflation", Federal Reserve Board, *Finance Econ. Discussion Paper no. 24*.
- Orphanides, A. and V. Wieland (1999), "Inflation zone targeting", European Central Bank, *Working paper no. 8*.
- Orphanides, A. and J.C. Williams (2002), "Imperfect Knowledge, Inflation Expectations and Monetary Policy", Board of Governors of the Federal Reserve System, *Working paper no.27*.
- Sargent, T.J. (1993), *Bounded Rationality in Macroeconomics*, Oxford University Press.
- Sargent, T.J. (1999), *The conquest of American inflation*, Princeton University Press.
- Simon, H.A. (1957), *Models of Man: Social and Rational*, New York, Wiley.
- Svensson, L.E.O. and M. Woodford (2000), "Indicator Variables for Optimal Policy", *NBER Working Paper no. 7953*.
- Wieland, V. (2003), "Monetary Policy and Uncertainty about the Natural Unemployment Rate", *CEPR Discussion Paper no. 3811*.
- Terlizzese, D. (1999), "A note on lexicographic ordering and monetary policy", mimeo.