Identifying business cycles for the Peruvian economy

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Introduction/Motivation

- Decomposition of GDP into cycle and trend component is an important statistical and theoretical problem for:
  - Study and analysis of recurrent expansions and recessions that occur in economic activity.
  - The design and implementation of macroeconomic policy.
- In other words, decomposing the GDP allows to infer the state or position of the economy relative to its long-term trend.

Definitions usually found in the academic literature and policy circles

- Econometrician point of view: Cycle-trend decomposition approach.
  - The trend is related to a stochastic trend.
  - The cycle is the deviation of GDP from its trend.
  - Beveridge-Nelson (1981), Clark (1986), Watson (1986) and Morley et al (2003).
- Policy maker point of view: Production function approach.
  - Potential output: level of GDP consistent with current technology and normal utilization of capital and labor input.
  - Output gap 2: Deviation of GDP.
  - Usually employed by CBO (2011) and many central banks.
- Macroeconomists (mostly academic) point of view: Natural equilibrium approach.
  - Deviations of GDP from the flexible-price equilibrium.
  - o Mostly related to the New Keynesian framework (Woodford (2003))

What do we do? 1

- We use an econometrics approach to decompose GDP into its cycle and trend component from 1980 to 2018 using an Unobserved Component model (UC) with a smoother trend component:
  - A stationary Cycle component.
  - Non-Stationary trend component. Specifically, the trend growth rate follows a random walk as in Chan and Grant (2017).
  - Unrestricted approach: Allow for correlation between cycle and trend innovations.
  - The model is labeled as UCUR-2M.
- Univariate approach: Quarterly data on GDP SA.

What do we do? 2

- Competing models: Chan and Grant (2017) show that the UCUR-2M model nests the standard HP filter as a special case by restricting the parameters to certain values.
  - UCUR-2M vs Nested set of HP models written in state-space (HP, HP-AR UC-2M).
  - UC models with deterministic breaks in the trend component.
- Bayesian estimation
  - o Falta resumen
- Model selection: Evaluation of the marginal likelihood and Bayes Factor
  - The marginal likelihood can be interpreted as the density forecast of the data under the model evaluated at the actual observed data. If the observed data are likely under the model, the associated marginal likelihood would be large.
  - Exact estimation of Bayes Factor.
- We use the best model to identify and date the Peruvian business cycle.

Literature review - Unobserved Component Framework

Transitry component: AR(2)

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \epsilon_t^c$$

Model	Permanent component	
Clark (1987)	$\tau_t = \mu_t + \tau_{t-1} + \epsilon_t^\tau$ $\mu_t = \mu_{t-1} + \epsilon_t^\mu$	$\begin{pmatrix} \epsilon_t^c \\ \epsilon_t^\tau \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_c^2 & 0 \\ 0 & \tilde{\sigma}_\tau^2 \end{pmatrix}\right)$
Morley (2003)	$\tau_t = \mu + \tau_{t-1} + \epsilon_t^\tau$	$\begin{pmatrix} \epsilon_t^c \\ \epsilon_t^\tau \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_c^2 & \rho\sigma_c\tilde{\sigma}_\tau \\ \rho\sigma_c\tilde{\sigma}_\tau & \tilde{\sigma}_\tau^2 \end{pmatrix}\right)$
UCUR- 1 Chan(2016)	$\tau_t = \mu_1 + \mu_2 + \tau_{t-1} + \epsilon_t^\tau$	
UCUR- 2 Chan(2016)	$\tau_t = \mu_1 + \mu_2 + \mu_3 + \tau_{t-1} + \epsilon_t^\tau$	
HP	$\triangle\tau_t = \triangle\tau_{t-1} + u_t^\tau \quad \lambda = 1600$	
HP + AR(2)	$\triangle\tau_t = \triangle\tau_{t-1} + u_t^\tau$ $\lambda = 1600$	$\begin{pmatrix} \epsilon_t^c \\ \epsilon_t^\tau \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_c^2 & 0 \\ 0 & \tilde{\sigma}_\tau^2 \end{pmatrix}\right)$
UC2M	$\triangle\tau_t = \triangle\tau_{t-1} + u_t^\tau$ $\hat{\lambda} = \frac{\sigma_c^2}{\tilde{\sigma}_\tau^2}$	
Perron and Wada (2009)	$\tau_t = \tau_{t-1} + \mu_t + \epsilon_t^\tau$ $\mu_t = \mu_{t-1} + \epsilon_t^\mu$	$\epsilon_t^\mu = \lambda_t\gamma_{1t} + (1-\lambda_t)\gamma_{2t}$
UCUR2M- Chan(2017)	$\triangle\tau_t = \triangle\tau_{t-1} + u_t^\tau$	$\begin{pmatrix} \epsilon_t^c \\ \epsilon_t^\tau \end{pmatrix} \sim N\left(0, \begin{pmatrix} \sigma_c^2 & \rho\sigma_c\tilde{\sigma}_\tau \\ \rho\sigma_c\tilde{\sigma}_\tau & \tilde{\sigma}_\tau^2 \end{pmatrix}\right)$

Empirical Strategy - UCUR2M

$$y_t = c_t + \tau_t \tag{1}$$

where τ_t is the trend and $c_t$is the stationariy ciclical component. The trend growth rate $(\tau_t)$is modeles as a random walk. whereas the ciclical component is modeled as a stationary AR(2) process with zero mean.

$$\triangle\tau_t = \triangle\tau_{t-1} + u_t^\tau \tag{2}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + u_t^c \tag{3}$$

The initial trend point τ_0 is treated as an unknow parameter. We assume innovation $u_t^c, u_t^\tau$are jointly normal.

$$\begin{pmatrix} u_t^c \\ u_t^\tau \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} \sigma_c^2 & \rho\sigma_c\sigma_\tau \\ \rho\sigma_c\sigma_\tau & \sigma_\tau^2 \end{pmatrix}\right) \tag{4}$$

First step

First of all , we have to rewrite our model in the following matrix form:

$$y = \tau + c$$

$$\mathbf{H}_\phi c = u^c$$

$$\mathbf{H}_2 \tau = \tilde{\alpha} + u^\tau$$

where $\tilde{\alpha} = (2\tau_0 - \tau_{-1}, -\tau_0, 0, \ldots, 0)'$ and

$$\mathbf{H}_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \end{pmatrix}, \ \mathbf{H}_\phi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\phi_1 & 1 & 0 & \cdots & 0 \\ -\phi_2 & -\phi_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\phi_2 & -\phi_1 & 1 \end{pmatrix}$$

both $\mathbf{H}_2$ and $\mathbf{H}_\phi$are band matrices with unit determinant , and therefore they are invertible.

Second step

To estimate the model in eq. (1), (2) and (3)., we construct a Gibbs sampler to sequentally sample the states $\tau = (\tau_1, ..., \tau_T)^{'}$ and the six parameters $\left(\phi, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}\right)$. *"The main challenge is the sampling of the states, which are fo dimension T. The states from linear Gaussian models are traditionally sampled by Kalman filter mased methods, such Carter and Kohn (1994), Früwirth-Schnatter (1994), Jong and Shepard (1995) and Durbin and Koopman (2002). To sample the states, we are going to use algorithms based on band matrix routines. One main advantage of this approach is the transparent derivation- all we need in standard linear regression... . In addition, these new algorithms are also more computationally efficient compared to Kalman filter based methods (Chan and Jeliazkov(2009))."*

Gibbs Sampler for the UCUR2M model

We can use the follwing 6-bolck Gibbs sampler to simulate from the joint posterior $p\left(\tau, \phi, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}/y\right)$. Pick some initial values $\phi^{(0)} \in \mathbb{R}, \sigma_c^{2(0)} > 0, \sigma_\tau^{2(0)} > 0, -1 < \rho^{(0)} < 1, \left(\tau_0^{(0)}, \tau_{-1}^{(0)}\right) = (y(1), y(1))$; where $\mathbb{R}$ is the stationay region.Then repeat the following setps from $r = 1$ to R

- Draw $\tau^{(r)} \sim p\left(\tau/y, \phi^{(r-1)}, \sigma_c^{2(r-1)}, \sigma_\tau^{2(r-1)}, \rho^{(r-1)}, \tau_0^{(r-1)}, \tau_{-1}^{(r-1)}\right)$ (multivariate normal)
- Draw $\phi^{(r)} \sim p\left(\sigma_c^2/y, \tau^{(r)}, \sigma_c^{2(r-1)}, \sigma_\tau^{2(r-1)}, \rho^{(r-1)}, \tau_0^{(r-1)}, \tau_{-1}^{(r-1)}\right)$ (normal)
- Draw $\sigma_c^{2(r)} \sim p\left(\sigma_c^2/y, \tau^{(r)}, \phi^{(r)}, \sigma_\tau^{2(r-1)}, \rho^{(r-1)}, \tau_0^{(r-1)}, \tau_{-1}^{(r-1)}\right)$ (uniform)
- Draw $\sigma_\tau^{2(r)} \sim p\left(\sigma_\tau^2/y, \tau^{(r)}, \phi^{(r)}, \sigma_c^{2(r)}, \rho^{(r-1)}, \tau_0^{(r-1)}, \tau_{-1}^{(r-1)}\right)$ (uniform)
- Draw $\rho^{(r)} \sim p\left(\sigma_\tau^2/y, \tau^{(r)}, \phi^{(r)}, \sigma_c^{2(r)}, \sigma_\tau^{2(r)}, \tau_0^{(r-1)}, \tau_{-1}^{(r-1)}\right)$ (uniform)
- Draw $\tau_0^{(r)}, \tau_{-1}^{(r)} \sim p\left(\sigma_\tau^2/y, \tau^{(r)}, \phi^{(r)}, \sigma_c^{2(r)}, \sigma_\tau^{2(r)}, \rho^{(r)}\right)$ (normal)

Chang's preposition

Preposition: Consider the unobserved components model with a second-order Markov transition defined by

$$y_t = c_t + \tau_t$$

$$\triangle\tau_t = \triangle\tau_{t-1} + u_t^{\tau}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + u_t^{c}$$

$$\begin{pmatrix} u_t^c \\ u_t^{\tau} \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} \sigma_c^2 & \rho\sigma_c\sigma_\tau \\ \rho\sigma_c\sigma_\tau & \sigma_\tau^2 \end{pmatrix}\right)$$

The Hodrick-Prescott trend $\hat{\tau}_{HP}$ is the posterior mean of τ by fixing: 1) Uncorrelated innovations between cycle and trend $\rho = 0$, and 3) the noise to signal ratio is constant $\left(\lambda = \frac{\sigma_C^2}{\sigma_\tau^2}\right)$ and 3) the cyclical components are serially uncorrelated $(\phi = 0)$.

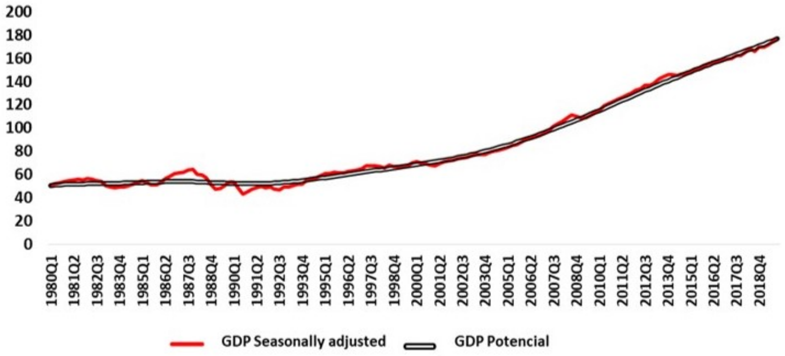
UCUR2M - Priors

Before carrying out the estimation process, the priors shown in the following table were used:

Parameters	Distributions	Hyperparameters
ϕ	Normal	$\mathcal{N}\left(\phi_{\mathbf{0}_{\mathbf{2X1}}}, \mathbf{V}_{\phi_{\mathbf{2X2}}}\right) 1\left(\phi \in \mathbb{R}\right);\ \phi_0 = [1.3; -0.4];\ V_\phi = I$
σ_c^2	Uniform	$U(0, b_c);\ b_c = 4.75$
σ_τ^2	Uniform	$U(0, b_\tau);\ b_\tau = 0.05$
ρ	Uniform	Bounded between -1 and 1.
τ_0 , τ_{00}	Normal	$\mathcal{N}\left(\tau_{00}, \mathbf{V}_\tau\right);\ \tau_{00} = 390;\ V_\tau = 100$

UCUR2M - Estimation

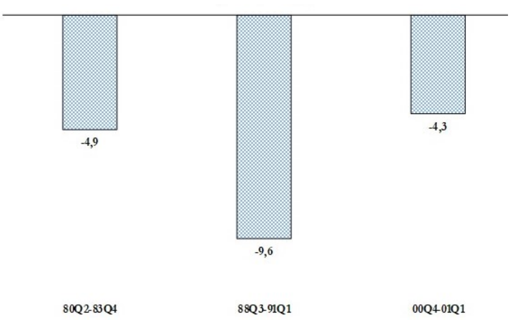
Figure: Gross Domestic Product - level and potential



Data - GDP growth rates

In Peru, there were events that were related to lower GDP growth rates. These events were associated with natural disasters, such as the El Nino Phenomenon of 1982-1983, as well as periods of hyperinflation in late 1980s and early 90's.The last event took place during the financial and political crisis of late 1990s and early 2000s.

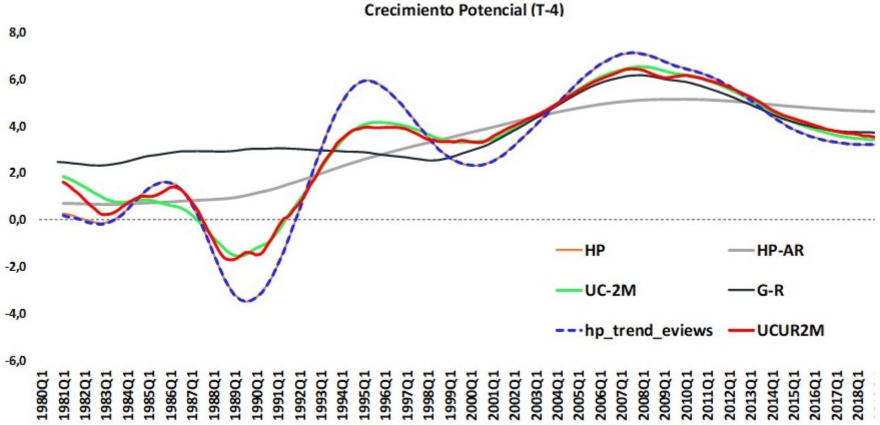
Figure: Real GDP Growth- quarterly average



The potential output growth rate

The following figure reports the growth rates of the potential output under the 5 methodologies analyzed.

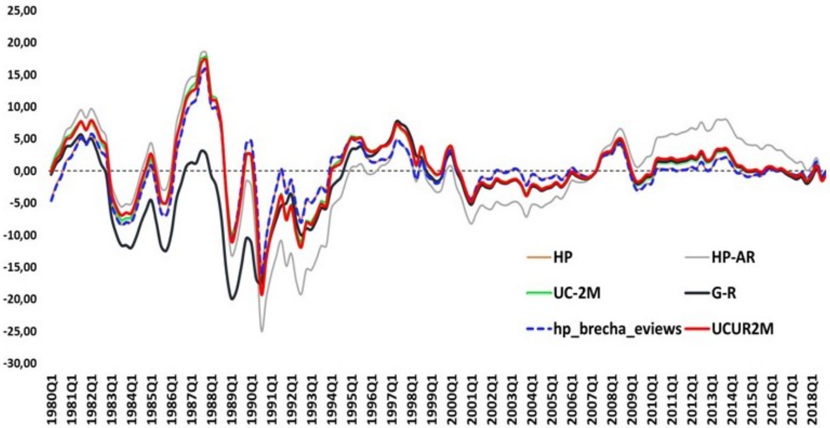
Figure: Real GDP Growth- YoY



The Output Gap

All methodologies report a highly volatile and wider output gap during 1980 and the late 1990s, with a subsequent moderation in terms of volatility and breadth during the last two decades of the analyzed period.

Figure: Output gap



Posteriors

The UC2M model, is the model that allows the lambda estimate as opposed to setting a predetermined value at 1600 as the HP model. The results show that the estimated lambda is very different from the default value for quarterly series.

Table: Posteriors

		UCUR2M	HP	HP-AR	UC-2M
ϕ_1	mean	1,26	-	1,33	1,28
	sd	0,10	-	0,07	0,07
ϕ_2	mean	-0,41	-	-0,41	-0,41
	sd	0,07	-	0,07	0,07
σ_c^2	mean	4,49	4,73	4,56	4,50
	sd	0,21	0,02	0,16	0,21
σ_τ^2	mean	0,03	-	-	0,03
	sd	0,01	-	-	0,01
ρ	mean	-0,23	-	-	-
	sd	0,56	-	-	-
$\hat{\lambda}$		-	-	-	129

Model selection: Evaluation of the integrated likelihood and Bayes Factor

The marginal likelihood can be interpreted as the density forecast of the data under the model evaluated at the actual observed data. If the observed data are likely under the model, the associated marginal likelihood would be large.

Table: Posteriors

	UCUR2M	**HP**	**HP-AR**	**UC-2M**
IL	-388,0	-709,3	-392,0	-388,8
sd	0,1	0,2	0,0	0,1

We use the best model to identify and date the Peruvian business cycle.

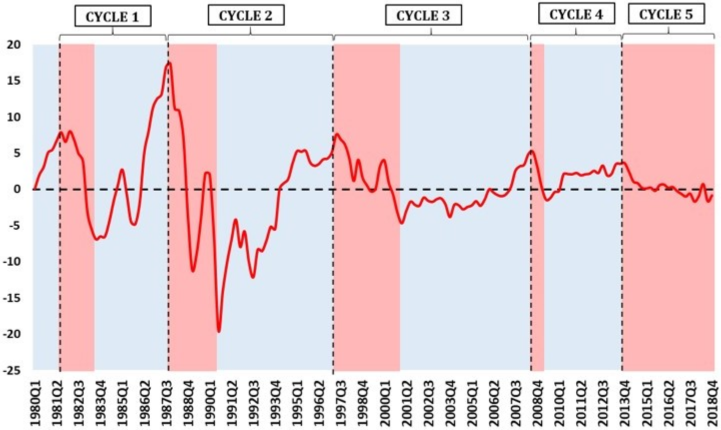
Identifying the Peruvian business cycle: Definitions

1. Peak: Local maximum for two consecutive previous and posterior quarters.
2. Trough: Local minimum for two consecutive previous and posterior quarters.
3. Amplitude: Vertical distance between a peak and a Trough.
4. Cycle: Distance between two peaks.
5. Contraction: Distance between a peak and trough.
6. Expansion: Distance between a trough and a peak.
7. Recession: Two consecutive quarters with negative GDP-SA growth.

Implications - Identifying Peruvian businees cycles

We identify 5 business cycles (4 complete cycles and the last one still in progress). The differences between the peaks and the valley of each economic cycle are called amplitude which is associated with their volatility.

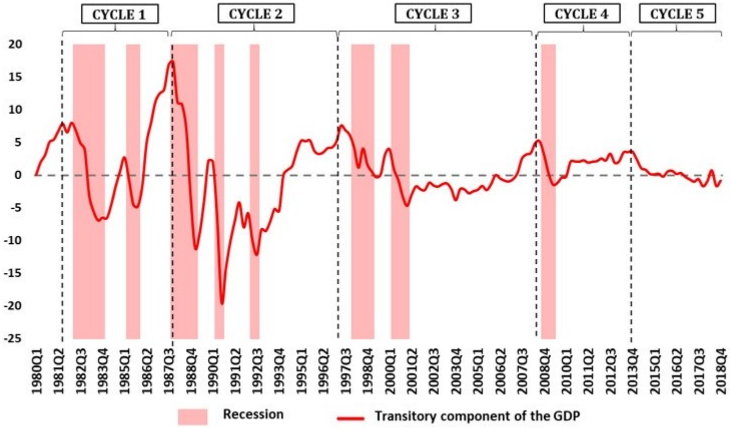
Figure: Business cycles and phases



Business cycles and Recessions

The following figure shows that the cyclical component registers falls during all recessions since 1980, where recession is defined as two consecutive quarters of negative growth of the seasonally adjusted GDP.

Figure: Business cycles and Recessions



Dating Peruvian business cycles

The four complete business cycles identified took place during 1981Q3-2013Q3 with a maximum duration of 45 quarters during the third business cycle and a minimum duration of 21 quarters during the fourth businees cycle.

Business Cycle					Recessions		
N°	Phase	Period	Duration	Maximun Amplitude	N°	Period	Duration
Cycle 1		**[1981Q3 - 1987Q3]**	**25**	**24,24**			
	Contraction	[1981Q3 - 1983Q2]	*8*	*14,78*	Recession 1	[1982Q2 - 1983Q4]	7
	Expansion	[1983Q3 - 1987Q3]	*17*	*24,24*	Recession 2	[1985Q2 - 1985Q4]	3
Cycle 2		**[1987Q4 - 1997Q1]**	**38**	**36,81**			
	Contraction	[1987Q4 - 1990Q2]	*11*	*36,81*	Recession 1	[1987Q4 - 1989Q1]	6
					Recession 2	[1990Q2 - 1990Q3]	2
	Expansion	[1990Q3 - 1997Q1]	*27*	*27,03*	Recession 3	[1992Q2 - 1992Q3]	2
Cycle 3		**[1997Q2 - 2008Q2]**	**45**	**12,25**			
	Contraction	[1997Q2 - 2000Q4]	*15*	*12,25*	Recession 1	[1998Q1 - 1999Q1]	5
					Recession 2	[2000Q2 - 2001Q1]	4
	Expansion	[2001Q1 - 2008Q2]	*30*	*9,86*			
Cycle 4		**[2008Q3 - 2013Q3]**	**21**	**6,60**			
	Contraction	[2008Q3 - 2009Q1]	*3*	*6,60*	Recession 1	[2008Q4 - 2009Q2]	3
	Expansion	[2009Q2 - 2013Q3]	*18*	*5,03*			
Cycle 5		**[2013Q4 -]**	**21**	**5,31**			
	Contraction	[2013Q4 -]	*21*	*5,31*			

Dating Peruvian business cycles and their main features

Table:

	Period	Growth rate	Output gap (average)	Potential output growth rate (average)
Contraction Periods				
Cycle 1	*[1981Q3 - 1983Q2]*	-1,6	3,7	0,7
Cycle 2	*[1987Q4 - 1990Q2]*	-5,7	1,5	-1,2
Cycle 3	*[1997Q2 - 2000Q4]*	2,4	2,4	3,5
Cycle 4	*[2008Q3 - 2009Q1]*	4,5	2,9	6,1
Cycle 5	*[2013Q4 -]*	3,4	0,2	4,1
Expansion Periods				
Cycle 1	*[1983Q3 - 1987Q3]*	4,4	1,9	0,9
Cycle 2	*[1990Q3 - 1997Q1]*	3,5	-2,9	2,3
Cycle 3	*[2001Q1 - 2008Q2]*	5,7	-1,1	5,2
Cycle 4	*[2009Q2 - 2013Q3]*	6,0	1,7	5,8

A decade-level analysis of the contribution to GDP growth rates

of the output gap and the potential output is shown in following table.

Table:

Real GDP growth
(contributions %)

	τ_t	c_t	y_t	GDP
1981-1990	0,2	-1,2	-1,0	-0,7
1991-2000	2,9	1,0	3,9	4,0
2001-2010	5,4	0,2	5,6	5,6
2011-2018	4,7	-0,4	4,3	4,3
1981-2018	**3,2**	**-0,1**	**3,1**	**3,2**

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Identifying business cycles for the Peruvian economy

ANNEX I - Estimation

Following Grant and Chan(2017) , under a Bayesian estimation method , we use a Markov sampler to obtain draws from the posterior distribution under the Unobserved Component Model. We assume proper but relatively noninformative priors for the model parameters $(\phi_1, \phi_2)\prime, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0$ andτ_{-1}. In particular, we assume indepent priors for ϕ, τ_0 and τ_{-1} :

$$\phi \sim \mathcal{N}\left(\phi_0, V_\phi\right) 1\left(\phi \in \mathbb{R}\right), \quad \tau_0, \tau_{-1} = \mathcal{N}\left(\tau_{00}, V_t\right)$$

where $\mathbb{R}$ is the stationary region. We assume relatively large prior variance $V_\phi = \mathbb{I}_2$ so that a priori ϕcan take on a wide of values. The prior is assumed to be $\phi_0 = 0.7, \tau_{00} = 390, V_\tau = 100$. Next , we assume the priors on $\sigma_c^2$and $\sigma_\tau^2$to be uniform:

$$\sigma_c^2 \sim \mathcal{N}\left(0, b_c\right), \ \sigma_\tau^2 \sim \mathcal{N}\left(0, b_\tau\right)$$

We have used a bayesian criteria to choose the value of b_c and $b_\tau(b_c = 4.75, b_\tau = 0.05)$.

Hence given the parameters $\phi, \sigma_c^2, \sigma_\tau^2$ and $\tau_0$we have:

$$\begin{pmatrix} c \\ \tau \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} \sigma_c^2 \left(\mathbf{H}_\phi'\mathbf{H}_\phi\right)^{-1} & \rho\sigma_c\sigma_\tau \left(\mathbf{H}_2'\mathbf{H}_\phi\right)^{-1} \\ \rho\sigma_c\sigma_\tau \left(\mathbf{H}_\phi'\mathbf{H}_2\right)^{-1} & \sigma_\tau^2 \left(\mathbf{H}_2'\mathbf{H}_2\right)^{-1} \end{pmatrix}\right)$$

where $\alpha = \mathbf{H}_2^{-1}\hat{\alpha}$. Using the properties of the Gaussian distributions , the marginal distribution of τ(unconditional on c) is:

$$(\tau \mid \sigma_\tau^2, \tau_0, \tau_{-1}) \sim \mathcal{N}\left(\alpha, \sigma_\tau^2 \left(\mathbf{H}_2'\mathbf{H}_2\right)^{-1}\right)$$

and the conditional distribution of y given τ and other parameters is given by:

$$(\mathbf{y} \mid \tau, \phi, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}) \sim \mathcal{N}\left(\mathbf{H}_\phi^{-1}\mathbf{a} + \mathbf{H}_\phi^{-1}\mathbf{B}\tau, \left(1-\rho^2\right)\sigma_c^2 \left(\mathbf{H}_\phi'\mathbf{H}_\phi\right)^{-1}\right)$$

where $\mathbf{a} = \frac{\rho\sigma_c}{\sigma_\tau}\mathbf{H}_2\alpha$ and $\mathbf{B} = \mathbf{H}_\phi + \frac{\rho\sigma_c}{\sigma_\tau}\mathbf{H}_2$

Thus, the prior density of τ and the conditional likelihood are given by:

$$p(\tau \mid \sigma_\tau^2, \tau_0, \tau_{-1}) = \left(2\pi\sigma_\tau^2\right)^{-\frac{T}{2}} \exp^{-\frac{1}{2\sigma_\tau^2}(\tau-\alpha)'\mathbf{H}_2'\mathbf{H}_2(\tau-\alpha)}$$

$$p(\mathbf{y} \mid \tau, \phi, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}) = \left(2\pi\sigma_y^2\left(1-\rho^2\right)\right)^{-\frac{T}{2}} \exp^{\frac{\left(\mathbf{H}_\phi\mathbf{y}-\mathbf{a}-\mathbf{B}\tau\right)'\left(\mathbf{H}_\phi\mathbf{y}-\mathbf{a}-\mathbf{B}\tau\right)}{2(1-\rho^2)\sigma_c^2}}$$

using the standar linear regression results,we get:

$$\left(\tau \mid y, \phi, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}\right) \sim \mathcal{N}\left(\widehat{\tau}, \mathbf{K}_\tau^{-1}\right)$$

where

$$\mathbf{K}_\tau = \frac{1}{\sigma_\tau^2}\mathbf{H}_2'\mathbf{H}_2 + \frac{1}{\left(1-\rho^2\right)\sigma_c^2}\mathbf{B}'\mathbf{B},$$

and

$$\widehat{\tau} = \mathbf{K}_\tau^{-1}\left(\frac{1}{\sigma_\tau^2}\mathbf{H}_2'\mathbf{H}_2\alpha + \frac{1}{\left(1-\rho^2\right)\sigma_c^2}\mathbf{B}'\left(\mathbf{H}_\phi\mathbf{y}-\mathbf{a}\right)\right)$$

since $\mathbf{H}_2$and $\mathbf{H}_\phi$band matrices, so $\mathbf{K}_\tau$ is a precision matrix. The precision sampler of Chan and Jeliazkov(2009) can be used efficiently.

To sample ϕin step 2, we can realize easily that $u^c$and τ are jointly normal:

$$\begin{pmatrix} u^c \\ \tau \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} \sigma_c^2 \mathbb{I}_{\mathbf{T}} & \rho\sigma_c\sigma_\tau \left(\mathbf{H}_2'\right)^{-1} \\ \rho\sigma_c\sigma_\tau \mathbf{H}_2^{-1} & \sigma_\tau^2 \left(\mathbf{H}_2'\mathbf{H}_2\right)^{-1} \end{pmatrix}\right)$$

recall that $\alpha = \mathbf{H}_2'^{-1}\widehat{\alpha}$.Hence, the conditional distribution of u^c given τ and other parameters is:

$$\left(\mathbf{u^c} \mid \tau, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}\right) \sim \mathcal{N}\left(\frac{\rho\sigma_c}{\sigma_\tau}\mathbf{H}_2\left(\tau - \alpha\right), \left(1 - \rho^2\right)\sigma_c^2\mathbb{I}_{\mathbf{T}}\right)$$

then, we rewrite eq(2) as:

$$c = \mathbf{X}_{\phi}\phi + u^{c}$$

Where $X_{\phi}$is a $Tx1$ matriz consisting of lagged value c_t.By the standar regression we get:

$$(\phi \mid y, \tau, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}) \sim \mathcal{N}\left(\hat{\phi}, \mathbf{K}_{\phi}^{-1}\right) 1 (\phi \in \mathbb{R})$$

Where :

$$\mathbf{K}_{\phi} = \mathbf{V}_{\phi}^{-1} + \frac{1}{(1-\rho^2)\,\sigma_c^2}\mathbf{X}_{\phi}'\mathbf{X}_{\phi}$$

and

$$\hat{\phi} = \mathbf{K}_{\phi}^{-1}\left(\mathbf{V}_{\phi}^{-1}\phi_0 + \frac{1}{(1-\rho^2)\,\sigma_c^2}\mathbf{X}_{\phi}'\left(c - \frac{\rho\sigma_c}{\sigma_\tau}\mathbf{H_2}(\tau-\alpha)\right)\right)$$

a draw form this truncated normal distribution can be obtained by the acceptance-rejection model (keep sampling until$\phi \in \mathbb{R}$).

To implement step 3 and 4, we first derive the joint density of $u^c$and u^τ . To that end, using σ_c^2andσ_τ^2, we get the joint distribution of (u^c, u^τ)as:

$$u^\tau \sim \mathcal{N}\left(0, \sigma_\tau^2\right) ; \left(u_t^c \mid u_t^\tau\right) \sim \mathcal{N}\left(\frac{\rho\sigma_c}{\sigma_\tau} u_t^\tau, \left(1-\rho^2\right)\sigma_c^2\right)$$

Hence, the joint density function of $u^c$and $u^\tau$is:

$$p(\mathbf{u^c}, \mathbf{u}^\tau \mid \sigma_c^2, \sigma_\tau^2, \rho) \propto \left(\sigma_\tau^2\right)^{\frac{-T}{2}} \exp^{\frac{\sum_{t=1}^T \left(u_t^\tau\right)^2}{2\sigma_\tau^2}} \left(\left(1-\rho^2\right)\sigma_c^2\right)^{-\frac{T}{2}} \exp^{\frac{\sum_{t=1}^T \left(u_t^c - \frac{\rho\sigma_c}{\sigma_\tau} u_t^\tau\right)^2}{2(1-\rho^2)\sigma_c^2}}$$

$$= \left(\left(1-\rho^2\right)\sigma_\tau^2\sigma_c^2\right)^{-\frac{T}{2}} \exp^{-\frac{1}{2\sigma_\tau^2}k_3 - \frac{1}{2(1-\rho^2)\sigma_c^2}\left(k_1 - \frac{2\rho\sigma_c}{\sigma_\tau}k_2 + \frac{\rho^2\sigma_c^2}{\sigma_\tau^2}k_3\right)}$$

Where $k_1 = \sum_{t=1}^T \left(u_t^c\right)^2$ $k_2 = \sum_{t=1}^T u_t^c u_t^\tau$and $k_3 = \sum_{t=1}^T \left(u_t^\tau\right)^2$. It follows from the latest equation that:

$$p(\sigma_c^2 \mid y, \tau, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}) \propto p\left(\sigma_c^2\right) \times \left(\sigma_c^2\right)^{-\frac{T}{2}} \exp^{-\frac{1}{2(1-\rho^2)\sigma_c^2}\left(k_1 - \frac{2\rho\sigma_c}{\sigma_\tau}k_2 + \frac{\rho^2\sigma_c^2}{\sigma_\tau^2}k_3\right)}$$

Where $p\left(\sigma_c^2\right)$is the prior for σ_c^2.

Similarly , we can implement step 4 and 5:

$$p(\sigma_\tau^2 \mid y, \tau, \sigma_c^2, \rho, \tau_0, \tau_{-1}) \propto p\left(\sigma_\tau^2\right) \times \left(\sigma_\tau^2\right)^{-\frac{T}{2}} \exp^{-\frac{1}{2\sigma_\tau^2}k_3 - \frac{k_1 - \frac{2\rho\sigma_c}{\sigma_\tau}k_2 + \frac{\rho^2\sigma_c^2}{\sigma_\tau^2}k_3}{2(1-\rho^2)\sigma_c^2}}$$

$$p(\rho \mid y, \tau, \sigma_c^2, \sigma_\tau^2, \tau_0, \tau_{-1}) \propto p\left(\rho\right) \times \left(1 - \rho^2\right)^{-\frac{T}{2}} \exp^{-\frac{k_1 - \frac{2\rho\sigma_c}{\sigma_\tau}k_2 + \frac{\rho^2\sigma_c^2}{\sigma_\tau^2}k_3}{2(1-\rho^2)\sigma_c^2}}$$

Where $p\left(\sigma_\tau^2\right)$and $p\left(\rho\right)$are the prior for σ_τ^2 and ρ, respectively. We have to mention that conditional densities for both of σ_c^2 and , which ones where obtaine in step 3 and 4, are not a standar densities and in order to sample from it, we use a Gridde-Gibbs step (evaluating the full conditional density on a fine grid and obtaining a draw from the density using the inverse-transform method-Kroese, Taimre and Botev, 2011).

Finally, recall that τ_0 we can write $\alpha = \mathbf{X}_\delta \delta$, where $\delta = (\tau_0, \tau_{-1})'$and:

$$\mathbf{X}_\delta = \begin{pmatrix} 2 & -1 \\ 3 & -2 \\ \vdots & \vdots \\ T+1 & -T \end{pmatrix}$$

$$\left(\tau \mid u^c, \sigma_c^2, \sigma_\tau^2, \rho, \tau_0, \tau_{-1}\right) \backsim \mathcal{N}\left(X_\delta \delta + \frac{\rho \sigma_\tau}{\sigma_c} \mathbf{H}_2^{-1} u^c, \left(1 - \rho^2\right) \sigma_\tau^2 \left(\mathbf{H}_2' \mathbf{H}_2\right)^{-1}\right)$$

Where $u_t^\tau \backsim \mathcal{N}\left(0, \sigma_\tau^2\right)$. Given the normal prior $\tau_0 \backsim \mathcal{N}\left(a_0, b_0\right)$, using the standar regression result , we get:

$$\left(\tau_0, \tau_{-1} \mid y, \tau, \phi, \sigma_c^2, \sigma_\tau^2, \tau_0, \tau_{-1}\right) \backsim \mathcal{N}\left(\hat{\delta}, \mathbf{K}_\delta^{-1}\right)$$

Where:

$$\mathbf{K}_\delta = \mathbf{V}_\delta^{-1} + \frac{1}{\left(1 - \rho^2\right) \sigma_\tau^2} \mathbf{X}_\delta' \mathbf{H}_\delta' \mathbf{X}_\delta \mathbf{H}_\delta$$

$$\hat{\delta}_0 = \mathbf{K}_\delta^{-1} \left(\mathbf{V}_\delta^{-1} \delta_0 + \frac{\tau_1}{\left(1 - \rho^2\right) \sigma_\tau^2} \left(\tau - \frac{\rho \sigma_\tau}{\sigma_c} \mathbf{H}_2^{-1} u^c\right)\right)$$

Where $\mathbf{V}_\delta = diag\left(V_\tau, V_\tau\right)$and $\delta_0 = \left(\tau_{00}, \tau_{00}\right)'$

ANNEX II - Integrated marginal likelihood

These integrated likelihoods can then be evaluated using band matrix routines, which are more efficient than using the conventional Kalman filter.

$$p\left(\tau/\sigma_\tau^2, \mu_1, \tau_0\right) = \left(2\pi\sigma_\tau^2\right)^{-\frac{T}{2}} \exp^{-\frac{1}{2\sigma_\tau^2}(\tau-\alpha)'H'H(\tau-\alpha)}$$

$$p\left(y/\tau, \phi, \sigma_c^2, \sigma_\tau^2, \rho, \mu_1, \tau_0\right) = \left(2\pi\sigma_c^2\left(1-\rho^2\right)\right)^{-\frac{T}{2}}$$

$$\times \exp^{-\frac{1}{2(1-\rho^2)\sigma_c^2}\left(H_\phi y - a - B\tau\right)'\left(H_\phi y - a - B\tau\right)}$$

where

$$a = -\frac{\rho\sigma_c}{\sigma_\tau}H\alpha, \;\; B = H_\phi + \frac{\rho\sigma_c}{\sigma_\tau}H$$

Let $k_4 = (2\pi)^{-T}\left(\left(1-\rho^2\right)\sigma_c^2\sigma_\tau^2\right)^{-\frac{T}{2}}$. Then the integrated likelihood can be derived as follows:

$$p\left(y/\phi,\sigma_c^2,\sigma_\tau^2,\rho,\mu_1,\tau_0\right)=\int p\left(y/\tau,\phi,\sigma_c^2,\sigma_\tau^2,\rho,\mu_1,\tau_0\right)p\left(\tau/\sigma_\tau^2,\mu_1,\tau_0\right)d\tau$$

$$=k_4\int \exp^{-\frac{1}{2(1-\rho^2)\sigma_c^2}\left(H_\phi y-a-B\tau\right)'\left(H_\phi y-a-B\tau\right)}\exp^{-\frac{1}{2\sigma_\tau^2}(\tau-\alpha)'H'H(\tau-\alpha)}d\tau$$

$$=k_4\times$$

$$\int \exp^{-\frac{1}{2}\left[\frac{1}{(1-\rho^2)\sigma_c^2}\left(H_\phi y-a-B\tau\right)'\left(H_\phi y-a-B\tau\right)+\frac{1}{\sigma_\tau^2}(\tau' H' H\tau-2\tau' H' H\alpha+\alpha' H' H\alpha)\right]}d\tau$$

$$=k_4\exp^{-\frac{1}{2}\left[\frac{1}{(1-\rho^2)\sigma_c^2}\left(H_\phi y-a\right)'\left(H_\phi y-a\right)+\frac{1}{\sigma_\tau^2}(\alpha' H' H\alpha)\right]}\times$$

$$\int \exp^{-\frac{1}{2}(\tau' K_\tau\tau-2\tau' d_\tau)}d\tau$$

$$= k_4 \exp^{-\frac{1}{2}\left[\frac{1}{(1-\rho^2)\sigma_c^2}\left(H_\phi y - a\right)'\left(H_\phi y - a\right) + \frac{1}{\sigma_\tau^2}(\alpha' H' H\alpha) - d_\tau' K_\tau^{-1} d_\tau\right]} \times$$

$$\int \exp^{-\frac{1}{2}\left(\tau - K_\tau^{-1} d_\tau\right)' K_t \left(\tau - K_\tau^{-1} d_\tau\right)} d\tau$$

$$= k_4 \exp^{-\frac{1}{2}\left[\frac{1}{(1-\rho^2)\sigma_c^2}\left(H_\phi y - a\right)'\left(H_\phi y - a\right) + \frac{1}{\sigma_\tau^2}(\alpha' H' H\alpha) - d_\tau' K_\tau^{-1} d_\tau\right]} \times (2\pi)^{\frac{T}{2}} \left|K_\tau\right|^{-\frac{1}{2}}$$

$$= \left(2\pi\left(1-\rho^2\right)\sigma_c^2\sigma_\tau^2\right)^{-\frac{T}{2}} \left|K_\tau\right|^{-\frac{1}{2}} \times$$

$$\exp^{-\frac{1}{2}\left[\frac{1}{(1-\rho^2)\sigma_c^2}\left(H_\phi y - a\right)'\left(H_\phi y - a\right) + \frac{1}{\sigma_\tau^2}(\alpha' H' H\alpha) - d_\tau' K_\tau^{-1} d_\tau\right]}$$

where:

$$K_\tau = \frac{1}{\sigma_\tau^2} H' H + \frac{1}{\left(1-\rho^2\right)\sigma_c^2} B' B, \quad d_\tau = \frac{1}{\sigma_\tau^2} H' H\alpha + \frac{1}{\left(1-\rho^2\right)\sigma_c^2} B'\left(H_\phi y - a\right)$$

since H, H_ϕ, and $K_\tau$are band matrices, this integrated likelihood can be evaluated quickly using the band matrix algorithms discussed in Chan and Grant (2016).