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Informality and Wealth Distribution: A Heterogeneous Agent Model

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Informality and Wealth Distribution: A Heterogeneous Agent Model*

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Abstract

We postulate a continuous-time heterogeneous agent model that incorporates four key characteristics of informality: high informality size, interest rate premium, exemption from taxes, and greater risk aversion of informal agents. We use this framework to study the implications of informality for wealth and consumption distribution. Our results align with empirical research, showing that a substantial informal sector reduces overall median wealth and consumption levels while increasing their dispersion. We also identify differentiated contributions to this result from each of the four features of informality. Greater informality size and higher risk aversion among informal agents raise wealth dispersion, while a higher interest rate premium among informal agents lessens this statistic. Informal tax evasion, on the other hand, has only minor impacts on these results. This model can be extended to provide insights for designing economic policies in emerging and developing countries.

Keywords: Informal employment, shadow economy, heterogeneous agents, wealth inequality
JEL: E10, E21, E26

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1 Introduction

The informality of the labor market is a prominent characteristic of emerging and developing countries. This is illustrated by the remarkable size of informal employment in Latin American countries, approximately 54%. This magnitude rises even further to 70% when considering a comprehensive measurement across emerging and developing nations.[1] Therefore, the design and evaluation of economic policies in these countries must consider this inherent structural aspect of their labor market.

Given the relevance of informality, the existing literature has focused on comprehending its causes and consequences (e.g., Ulyssea, 2020). The theoretical branch of the literature has focused on examining the *average effect* of informality on aggregate variables such as trade, tax collection, productivity, and economic development (Leal Ordóñez, 2014; Almeida and Poole, 2017; Dellas et al., 2024). Meanwhile, the empirical side has aimed to shed light on the disparities between formal and informal employees. Specifically, empirical studies have consistently shown that informal workers tend to experience lower income, higher income volatility, lack of tax payment, and greater risk aversion compared to their formal counterparts (e.g., Maya and Pereira, 2020; Gomes et al., 2020; Wang, 2022).

Despite considerable efforts to understand the repercussions of informality, the knowledge regarding its effects on wealth and consumption distribution remains limited. The importance of filling this knowledge gap is twofold. First, understanding the distributional effects of informality contributes to comprehending its role in inequality and the underlying economic mechanisms. Second, a comprehensive understanding of the distributional impacts of informality is crucial to designing effective economic policies. Given that these policies may have heterogeneous effects on both formal and informal workers, as well as on the wealth and consumption distribution of these agents, understanding the role of informality in shaping these distributions emerges as a primary concern.

This paper aims to address the aforementioned gap by investigating the distributional effects of informality, an aspect surprisingly absent in the current literature. Specifically, we explore how informality influences wealth and consumption distributions. This investigation is motivated by the need for a framework that incorporates the main characteristics of informality and allows for the evaluation of policy effects on the distribution of macroeconomic variables, thereby facilitating the assessment of different policies in developing countries. While our framework does not directly tackle the causes of informality, it provides a means to evaluate policies that reduce the probability of individuals working informally and, consequently, their distributional effects.

To achieve this objective, we develop a quantitative heterogeneous agent model based on the Bewley-Imrohoroglu-Huggett-Aiyagari incomplete market framework, a leading building block in modern macroeconomics (see Bewley, 1977; Imrohoroglu, 1989; Huggett, 1993; Aiyagari, 1994). Specifically, we build upon Achdou et al. (2022)'s continuous-time version of Huggett's model, incorporating a crucial extension: the inclusion of an informal agent characterized by four key attributes —low income, non-payment of taxes, an interest rate premium, and higher risk aversion. Previous studies have found that these characteristics effectively typify an informal worker (see for example Bennett et al., 2012; Horvath, 2018; Gomes et al., 2020). First, informal employment does not require the employer to make social security contributions or the employee's income subject to tax collection. Second, borrowing interest rates paid by more informal sectors and countries are higher than by less informal ones. Lastly, the decision to become a self-employed or an informal worker is, on average, associated with a greater risk aversion of workers. These elements introduce structural heterogeneities among agents, enabling us to study the distributional effects of informality and providing a framework for policy evaluation.

Our primary finding reveals that the informal economy, with these aforementioned features, ex-

[1] These numbers are based on 2018 statistics of the informal economy calculated by the International Labor Organization. For further details, see "*Women and men in the informal economy: a statistical picture (third edition) / International Labour Office – Geneva: ILO, 2018*"

hibits wealth and consumption distributions with lower median and higher dispersion when compared to a benchmark economy–an economy with *only* a low informality *size* level. Regarding wealth distribution, the informal population is leveraged primarily due to their low income and reliance on borrowing for consumption financing, while the formal population has a higher median wealth. In contrast, the formal population exhibits higher dispersion than their informal counterparts. These results suggest that the informal economy increases inequality and has spillover effects on formal agents.

A second important finding is that each characteristic of informal workers has distinct effects, both in magnitude and direction, on the wealth and consumption distribution. The positive difference in wealth's standard deviation between the informal and benchmark economy is primarily explained by the large size of informality and high-risk aversion. Concurrently, the interest rate premium reduces this statistic, while the absence of tax payment has a negligible impact. Similar patterns emerge when explaining the decrease in median wealth: informality size and risk aversion contribute to reducing the median, whereas the interest rate premium has the opposite effect. The identification of the marginal effects from these informality features improves our understanding of the informal economy's contribution to the economic dynamics.

Our third finding is that these informality characteristics also have heterogeneous effects on the wealth and consumption distribution of each specific group (formal and informal agents). Specifically, the absence of tax payment by informal agents increases their median consumption but slightly reduces it for formal agents. Furthermore, the interest premium paid by informal agents reduces the median and dispersion of their consumption but increases them for formal agents. These effects highlight the potential economic mechanisms associated with every characteristic of informality.

Our framework can potentially serve as a valuable tool for studying economic policies due to its ability to generate the distribution of endogenous variables for the entire economy, including both formal and informal agents. Moreover, this framework allows for a quantitative analysis of the marginal effects of various informality characteristics. However, to enhance its applicability for policy analysis, it may be necessary to introduce nominal rigidities and other frictions, similar to the approach taken by Kaplan et al. (2018) and Fernández-Villaverde et al. (2023), albeit without considering informality. This is a natural extension of our model, which we leave for future research.

This paper contributes to at least two strands of the literature. First, we contribute to the literature that quantifies the aggregate effects of informal markets in the economy (e.g., Shapiro, 2014; Restrepo-Echavarria, 2014; Fernández and Meza, 2015; Dix-Carneiro et al., 2021; Colombo et al., 2019; Salinas, 2021; Leyva and Urrutia, 2023; Lahcen, 2020; Gomez Ospina, 2023; Dellas et al., 2024). These studies typically employ a two-sector modeling approach, where one sector represents informal agents and the other represents formal agents. While exploring labor market frictions, Shapiro (2014) focuses on self-employment, and Restrepo-Echavarria (2014) investigate informality in an open economy. Similarly, Dix-Carneiro et al. (2021), Colombo et al. (2019), and Salinas (2021) develop structural equilibrium models involving formal and informal firms, while Gomez Ospina (2023) and Dellas et al. (2024) study informality and monetary and fiscal policies. However, these studies do not adequately address the distributional effects of informality. Our paper aims to fill this gap by presenting a heterogeneous agent model that analyzes the effects of informality on the distribution of wealth and consumption in equilibrium. Additionally, our model incorporates a comprehensive and explicit treatment of heterogeneities among formal and informal workers, which is missing in previous research.

Our paper is also related to the literature that examines heterogeneous agents with incomplete markets in the spirit of Bewley (1977), Imrohoroglu (1989), Huggett (1993), and Aiyagari (1994). This growing literature has explored important topics such as precautionary savings, income uncertainty, wealth inequality, and monetary policy (e.g., Kaplan and Violante, 2014; Kaplan et al., 2018; Achdou et al., 2022; Fernández-Villaverde et al., 2023). However, despite the relevance of informality in developing countries, informality has not been explored within this framework to the best of our knowledge. In our paper, we contribute to this literature by incorporating the four key characteristics of informal agents into a heterogeneous agent model. Our objectives are twofold: to investigate the distributional effects of informality and to provide a framework that explicitly considers informality for policy analysis.

Finally, we solve our continuous-time model utilizing the finite difference method, a numerical technique described by Achdou et al. (2022). Following the suggestion by Kaplan et al. (2018), the continuous-time framework offers computational advantages and provides a parsimonious setup for modeling individual earnings with heterogeneities among agents.

The paper is structured as follows: Section 2 presents the heterogeneous agent model incorporating informality. In Section 3, we calibrate and solve the model numerically. Next, in Section 4, we conduct model simulations to analyze the impact of each informality characteristic on the distribution of wealth and consumption. Section 5 delves into the marginal contribution of each characteristic. Finally, Section 6 provides concluding remarks.

2 The Economic Model

We build a continuous-time heterogeneous agent model that encompasses both formal and informal workers within a single economy. Our model incorporates the realistic dynamics observed in the data, allowing agents to transition between formal and informal employment states with an exogenous probability, as documented by (Gomes et al., 2020). Specifically, we characterize informal agents as individuals with lower incomes compared to formal workers, who do not pay taxes, face an interest rate premium when borrowing, and exhibit higher levels of risk aversion—characteristics that are empirically supported (e.g., Gomes, 2020; Wang, 2022). To develop our model, we elaborate on the continuous-time general equilibrium model with incomplete markets and uninsured labor idiosyncratic risk proposed by Achdou et al. (2022), which is closely related to Bewley (1977), Imrohoroglu (1989), Huggett (1993), and Aiyagari (1994).

To examine the effects of informality on wealth distribution, it is crucial to establish a benchmark economy for comparison. Ideally, the benchmark economy would consist solely of formal workers, representing a formal economy. However, in such a scenario, the economy reduces to a representative agent framework, which lacks the necessary components for meaningful comparisons with the informal economy, such as consumption and wealth distributions. To address this limitation, we construct the benchmark economy with a small degree of informality. In the benchmark economy, formal and informal workers primarily differ in income[2], and informal workers have lower incomes than their formal counterparts. This allows us to effectively analyze and contrast the wealth distribution between the formal and informal sectors.

2.1 The Formal and Informal Economy

Before delving into the details of our model (sections 2.2 and 2.3), it is important to outline the key characteristics of the formal and informal economies. Our framework considers a diverse range of agents who vary in their wealth a and income y. In addition, these agents differ in some structural parameters that include the tax rate τ, interest rate premium θ, and risk attitude γ. With this continuum of heterogeneous agents, we define the formal and informal economies as follows.

[2]The other characteristics such as tax payment, interest rate for borrowing, and risk-aversion attitude are the same for formal and informal agents.

Formal economy (benchmark). We start with a *benchmark economy* characterized by informal agents having *lower* income than formal agents. In this economy: (i) there is a small informality size $\eta_0 = 20\%$, (ii) formal and informal agents pay taxes, (iii) both have the same borrowing interest rate, and (iv) both have the same risk-aversion parameter.

Informal economy. The informal economy consists of: (i) a high informality size $(\eta > \eta_0)$, informal agents (ii) do not pay taxes (Gomes et al., 2020), (iii) pay a higher borrowing interest rate (risk premium) (Horvath, 2018), and (iv) are more risk-averse (Bennett et al., 2012). Therefore, the informal economy is characterized by

1. *Informality size.* A high informality size compared to the benchmark economy: $\eta > \eta_0$,

2. *Taxes.* Informal agents do *not pay* income taxes: $\tau_2 > \tau_1 = 0$,

3. *Interest rate.* Informal agents pay a *premium* θ when they borrow. The informal worker pays $R = r + \theta$, while the formal worker pays $R = r$ for borrowing,

4. *Risk aversion.* The informal agent is *more risk-averse* than the formal agent: $\gamma_1 > \gamma_2$.

In our simulations (Section 4), we progressively add these four characteristics of the informal economy to the benchmark. We first increase the informality size ($\eta = 0.64$), then we consider non-pay taxes by the informal agent ($\tau_1 = 0$). We next include the premium interest rate ($\theta = 2\%$) and finally a higher relative risk aversion ($\gamma_1 = 0.3 > \gamma_2 = 0.15$). For every new informality characteristic added to the model, we compare the statistics of wealth and consumption distribution of the informal economy with the benchmark one and with the model without that characteristic. These comparisons shed light on the distributional effects of informality in the whole economy and the marginal effects of every informality characteristic.

2.2 The Economic Setting

2.2.1 Goods Markets

We assume an endowment economy in which the agent's income is exogenously determined. Furthermore, the consumption space C_+ is defined as the set of positive, adapted consumption rate processes c that satisfy the integrability condition

$$\int_0^T c_{it}^2 dt \leq \infty, \tag{1}$$

for every agent i in the economy.

2.2.2 Financial Markets

The financial market is incomplete due to the lack of insurance of the income risk the agent faces. Furthermore, the investment opportunity is represented by an unproductive bond in fixed net supply. The dynamic of the bond's price is modeled as follows:

$$dB_t = r_t B_t dt, \tag{2}$$

where r_t is the instantaneous interest rate.

2.2.3 Agents

A continuum of two types of agents populates the economy: the informal and the formal worker. The informal agent differs from the formal one in that he has a lower income, does not pay taxes, pays an interest rate premium, and is more risk-averse. It is also important to mention that heterogeneity in wealth exists among the sets of every type of agent. Consequently, our economy is characterized by wealth and consumption distribution for informal, formal, and total agents–a crucial difference from the standard informality literature that assumes two-agent types but *without* heterogeneity within every population (e.g., Restrepo-Echavarria, 2014; Shapiro, 2014; Dix-Carneiro et al., 2021; Colombo et al., 2019; Salinas, 2021). This typical modeling approach is equivalent to an economy with two representative agents (informal and formal), which unfortunately cannot generate wealth and consumption distributions. Our model, however, does not exhibit this weakness. This section describes agents' preferences, wealth dynamics, income process, and constraints.

Preferences. The two-type agents are heterogeneous in preferences, represented by a constant relative risk aversion (CRRA) utility function as follows.

$$U(c_{jt}, gov_t) = \frac{c_{jt}^{1-\gamma_j}}{1-\gamma_j} + gov_t, \quad j = 1, 2, \quad \gamma_1 > \gamma_2, \tag{3}$$

where agent $j = 1$ represents the informal worker and agent $j = 2$ represents the formal one. Furthermore, c_{jt} is the consumption rate of the agent j, and γ_j is his relative risk aversion parameter. We argue that the informal agent is more risk averse than the formal one ($\gamma_1 > \gamma_2$) mainly for two reasons. First, it is well-known in asset pricing literature that risk aversion declines when wealth increases. Considering this fact in our model, the informal worker has less income and less wealth; then it is expected that he is more risk averse. Second, given the labor income risk in our economy, and the low income of the informal agent, he would be less willing to substitute future consumption for current consumption; i.e., he would have a low elasticity of intertemporal substitution or equivalently a high relative risk aversion (RRA) (due to CRRA utility function).

We also assume a government exists and uses taxes to create public goods gov_t, which increase the agent's welfare equally among agents, independent of their types. We operationalize these assumptions by introducing an additive term, gov_t, in the agent's utility function. It is worth mentioning that the role of public goods does not change the equilibrium.

Agent type and income process. A crucial empirical difference between the informal and formal agents is the income level: the informal agent's income is lower than the formal one (e.g., Gomes et al., 2020). We capture these two-agent types by modeling a two-state income process. We follow this strategy by modeling the agent's income as a two-state Poisson process: $y_t \in \{y_1, y_2\}$ with $y_1 < y_2$, in which y_1 represents the income of the informal worker and y_2 the income of the formal one. Because the informal worker is characterized not only by lower income but also by three additional structural characteristics (no taxes, interest rate premium, and high-risk aversion), the probability of jumping between these two income levels can be interpreted as the probability of jumping between informal and formal states, that is, being an informal or formal agent. Furthermore, the probability of changing states is associated with an intensity parameter λ. Specifically, the income jumps from state 1 (informal) to state 2 (formal) with intensity λ_1 and vice versa with intensity λ_2.

Figure 1 illustrates the two-state income process and its relationship with the two-type agent. If an agent in period t has a level income y_1 (i.e., he is an informal worker), he has a (conditional) probability P_{11} to have the same level of income in the next period $t + \Delta$ (i.e., he stays in the informality state).

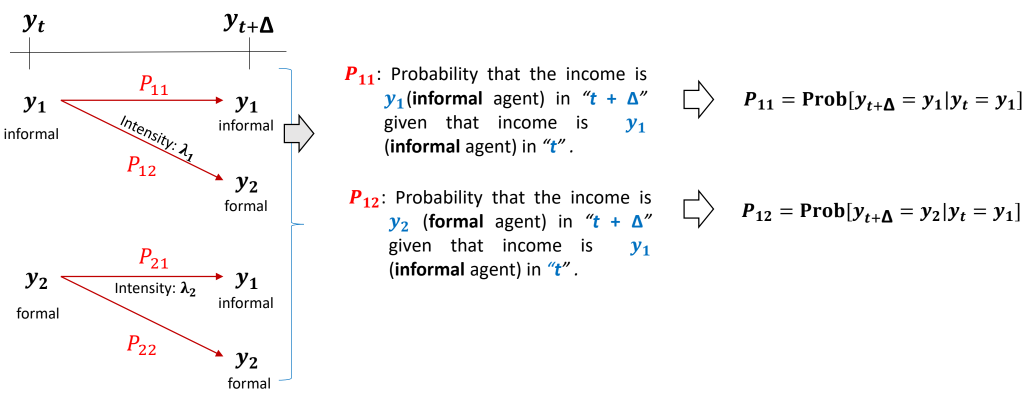

Figure 1
The Income Process: The Informal and Formal Agent

Then, P_{11} is defined by

$$P_{11} = \text{Prob}[y_{t+\Delta} = y_1 \mid y_t = y_1] \tag{4}$$

Similarly, P_{12} is the probability that an agent with income y_1 in period t (i.e., he is an informal worker) will have an income y_2 in period $t + \Delta$ (i.e., he is a formal worker). Then, P_{12} is defined by

$$P_{12} = \text{Prob}[y_{t+\Delta t} = y_2 \mid y_t = y_1] \tag{5}$$

The same intuition applied for P_{21} and P_{22}, in which the initial state is to be a formal agent. The intensity parameters λ_1 and λ_2 are connected with the previous conditional probabilities. We illustrate that connection assuming the agent is informal at t (the same applies if we consider the agent a formal worker at t). Next, for the informal agent at t (i.e., he has an income level of y_1), the probability to be a formal agent at $t + \Delta$ (i.e., he would have an income level of y_2) is given by

$$\begin{aligned} \text{Prob}[y_{t+\Delta} = y_2 \mid y_t = y_1] &= \lambda_1 \Delta + o(\Delta) && (6) \\ \text{Prob}[y_{t+\Delta} = y_1 \mid y_t = y_1] &= 1 - \lambda_1 \Delta + o(\Delta), && (7) \end{aligned}$$

where λ_1 is the intensity to jump from y_1 to y_2 or equivalently from being informal to being a formal worker, and $o(\Delta)$ is the asymptotic order symbol defined by

$$\frac{o(\Delta)}{\Delta} \to 0 \quad \text{when} \quad \Delta \to 0 \tag{8}$$

It is worth noting that the intensity parameters, λ_1 and λ_2, and the income levels, y_1 and y_2, are exogenous and will be calibrated. As Δ becomes infinitesimally small, probabilities P_{11} and P_{22} approach:

$$\begin{aligned} P_{11} &= e^{-\lambda_1}, \quad P_{12} = 1 - P_{11} && (9) \\ P_{22} &= e^{-\lambda_2}, \quad P_{21} = 1 - P_{22} && (10) \end{aligned}$$

Wealth dynamic. The investment opportunity in this economy is represented by the riskless asset (bond) with instantaneous interest rate r_t. Three forces drive the change in the agent's wealth: his income y_{jt}, his savings in the riskless asset a_{jt}, and his consumption c_{jt}. Therefore, the wealth dynamic of agent j is given by

$$da_{jt} = ((1 - \tau_j)\, y_{jt} + R_{jt} a_{jt} - c_{jt})\, dt, \quad j = 1, 2, \tag{11}$$

where $R_{2t} = r_t$ for the formal agent, while $R_{1t} = r_t + \theta$ for the informal agent when he is a borrower. The interest rate premium that the informal agent pays is $\theta > 0$, and the tax rate is τ_j. In our benchmark economy, both agents pay the same tax rate; however, in our informal economy, $\tau_1 = 0$ and $\tau_2 > 0$.

Borrowing constraint. We assume that informal and formal agents face the same borrowing limit $\underline{a}$. It is reasonable to think that $\underline{a}$ could differ between informal and formal agents. Specifically, the informal agent could have a more restricted borrowing limit than the formal one. We leave that analysis for future research. Then, we assume the following.

$$a_{jt} \geq \underline{a}, \quad \text{with } -\infty < \underline{a} < 0, \quad j = 1, 2. \tag{12}$$

The agent's optimization problem. The stochastic optimal control problem of the agent $j = 1, 2$, **P**, is defined as

$$
\begin{aligned}
\max_{\{c_{jt}\}} \quad & E_t\left[\int_0^{\infty} e^{-\rho t} U(c_{jt}, gov_t) dt\right] && (13)\\
\text{subject to} \quad & && \\
\text{Agent wealth dynamic} \quad : \quad & da_{jt} = \left((1-\tau_j)\, y_{jt} + R_{jt} a_{jt} - c_{jt}\right) dt && (14)\\
\text{Borrowing constraint} \quad : \quad & a_{jt} \geq \underline{a} && (15)\\
\text{Income process} \quad : \quad & y_{jt} \in \{y_{1t}, y_{2t}\} \text{ with } \lambda_1, \lambda_2 \text{ and } y_{1t} < y_{2t} && (16)\\
\text{Risk aversion} \quad : \quad & \gamma_j \in \{\gamma_1, \gamma_2\} \text{ with } \gamma_1 > \gamma_2 && (17)\\
\text{Tax rate} \quad : \quad & \tau_j \in \{\tau_1 = 0, \tau_2 > 0\} && (18)\\
\text{Interest rate} \quad : \quad & \text{Informal : if } a_{1t} < 0 \rightarrow R_{1t} = r_t + \theta && (19)\\
& \text{Formal : if } a_{2t} < 0 \rightarrow R_{2t} = r_t &&
\end{aligned}
$$

All agents take as given the interest rate $r_t, \forall t \geq 0$, to solve this problem. We solve this problem using the dynamic programming approach as Achdou et al. (2022).

2.3 Equilibrium

We follow Huggett (1993) and Achdou et al. (2022) to define and find the equilibrium.

2.3.1 Equilibrium Definition

We define equilibrium in the economy with informal and formal agents as follows.

Definition 2.1. Equilibrium in this economy is defined as consumption processes (c_{1t}, c_{2t}) and a price system (r) such that at every period t: (i) agents maximize their expected discounted utility function taking as given the equilibrium interest rate; i.e., they solve the optimization problem **P** (Eq. 13 - 19), and (ii) all markets (bonds and goods market) clear. The bond market equilibrium condition is given by

$$\underbrace{S(r)}_{\substack{\text{Total bonds}\\ \text{demand}}} \equiv \underbrace{\int_{\underline{a}}^{\infty} a dG_1(a,t)}_{\substack{\text{bonds demand from agents}\\ \text{who have income } y_1}} + \underbrace{\int_{\underline{a}}^{\infty} a dG_2(a,t)}_{\substack{\text{bonds demand from agents}\\ \text{who have income } y_2}} = \underbrace{B}_{\substack{\text{fixed}\\ \text{bonds supply}}}, \tag{20}$$

where the aggregate bond demand is represented by $S(r)$ and the aggregate supply is fixed and equals B. We assume the bond is in zero net supply and then $B = 0$. Furthermore, $G_j(a,t)$ is the cumulative distribution function (CDF) for agent type j at period t, and $d\tilde{G}_j(a) = g_j(a)da$. Furthermore, $g_j(a,t)$ represents the density of the *joint distribution* of y_j and a (in period t). The equilibrium condition for the second market, the goods market, is given by

$$c_{jt} + s_{jt} = y_{jt}, \quad j = 1, 2, \tag{21}$$

where s_{jt} is the saving of agent j and it is equals to the change of his wealth da/dt. Lastly, it is important to mention the budget constraint of the government, which is given by

$$\int \tau_1 y_1 \times g_1(a)da + \int \tau_2 y_2 \times g_2(a)da = \int gov \times g(a)d(a). \tag{22}$$

2.3.2 Finding the Equilibrium

We then follow the Achdou et al. (2022)'s strategy to find the equilibrium. First, the optimization problem of agents (from the equation (13) until the equation (19)) can be written recursively following the Bellman approach in continuous time. As a result, that optimization problem becomes a *system of partial differential equations (PDEs)* described by

1. *The Hamilton-Jacobi-Bellman (HJB) equation*

$$\rho V_j(a) = \max_{\{c\}} \{U(c_j, gov) + V_j'(a)S_j(a) + \lambda_j \left(V_{-j}(a) - V_j(a)\right)\} \tag{23}$$

2. *The Fokker-Planck (FP) equation*

$$0 = \partial_a[S_j(a)g_j(a,t)] - \lambda_j g(a,t) + \lambda_{-j} g_{-j}(a,t), \tag{24}$$

with the following conditions.

1. *The first-order conditions*
   (a) From HJB equation: $c_j(a) = (U')^{-1}(V_j'(a))$
   (b) Saving: $s_j(a) = y_j + Ra - c_j(a)$
2. *The state constraint boundary condition*

$$V_j'(\underline{a}) \geq U'(y_j + R\underline{a}) \tag{25}$$

3. *Interest rate*

$$\begin{aligned} \text{Informal agent} &: \quad \text{if } a < 0 \rightarrow R = r + \theta \\ \text{Formal agent} &: \quad \text{if } a < 0 \rightarrow R = r \end{aligned}$$

4. *The market clearing condition*

$$S(r) \equiv \int_{\underline{a}}^{\infty} a dG_1(a) + \int_{\underline{a}}^{\infty} a dG_2(a) = B \tag{26}$$

5. *The aggregation of distributions*

$$\int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da = 1 \tag{27}$$

Second, we use the finite difference method, which consists of approximating the first and second derivatives of the value function $V_j(a)$. We closely follow Achdou et al. (2022) in implementing this numerical method. Details of the use of this numerical method to find the equilibrium are provided in the following section.

3 The Numerical Solution Method

3.1 Calibration

In this section, we calibrate the model's parameters using standard values from heterogeneous agent literature, specifically from Peru, a developing country with a high level of informality.

We start with similar value parameters for the informal and formal agents. The first one is the subjective impatience rate $\rho = 0.05$, which is chosen to generate a discount factor equal to 0.95 (e.g., Chan and Kogan, 2002). The second one is the borrowing limit $\underline{a} < 0$, which is assumed to be equal to 30% (in absolute terms) of the minimum between the income of the informal and formal agents. We assume that the lender can observe the agent's life cycle and see both incomes (when he is informal and formal), then a risk-averse lender decides to offer a borrowing limit as the minimum of both. The value $\underline{a} = -30\% \min\{y_1, y_2\}$ is close to the upper bound of the debt-to-income ratio that lenders consider to offer a mortgage as a general guideline.[3] In general, the value of $\underline{a}$ is left to be arbitrary for numerical simulations.

The income level y. Using the annual income data of the Peruvian economy spanning from 2007 to 2022, we estimate that informal workers earn approximately one-third of the income earned by formal workers, on average. To facilitate comparative analysis, we normalize the labor income for the formal workers as $y_2 = 1$, which implies that the normalized income for informal workers is $y_1 = 0.33$.

The tax rate τ. The tax rate for informal workers is set at zero ($\tau_1 = 0$) because they do not pay income taxes. For formal workers, we calibrate their tax rate to be 0.18, representing the average income tax rate in Peru. This value is determined by calculating the simple average income tax rate for formal workers in the fourth (independent work) and fifth (dependent work) categories for income tax payments in Peru between 2016 and 2022.

The interest rate premium θ. The interest rate premium is zero for formal workers ($\theta_2 = 0$). However, it is positive for informal workers ($\theta_1 > 0$), which captures informal agents' frictions in accessing the financial system. To calculate θ_1, we use Peruvian monthly data from January 2015 to December 2019 on the interest rate of consumer loans from banks which usually lend to formal workers and from lenders oriented to informal workers called *saving agencies* (*cajas de ahorros* in Spanish). We find that, on average, informal agents pay 50% more than formal agents, in interest rate terms. We capture this behavior by assuming an interest rate premium θ_1 as 50% of the maximum interest rate in our model (4%). As a result, $\theta_1 = 0.02$.

The informal sector size η and RRA γ. We set the informal sector size, η, to 0.64, which is the average percentage of informal workers in the Peruvian economy during the period from 2011 to 2020. However, in the numerical analysis, we use a vector of informality size from $\eta = 0.2$ until $\eta = 0.9$ to evaluate their effects of equilibrium. Regarding the risk aversion parameter γ, we assume that the informal workers are twice as risk-averse as formal workers. We justify the higher risk-averse parameter for the informal agent based on the asset pricing fact, which states, "more wealth, the agent is less risk-averse." In our model, the formal agent has more wealth because he has more income and is less risk-averse.

[3] This is a general rule-of-thumb in banking sector illustrated by NerdWallet, a personal finance firm, in the following article.

Intensity λ. Regarding the intensity to jump between states, we estimate the transition matrix of the conditional probabilities to change or maintain in one of the two states (formal and informal) between two periods using annual data from 2011 to 2020 for Peru. We estimate the transition matrix using the square error minimization from a bivariate VAR, in which the two variables are the marginal probabilities of being formal and informal workers. It is important to mention that we assume that these marginal probabilities are the fraction of formal and informal workers in the Peruvian economy. Moreover, once we have the conditional probabilities, we use the following transformation to find the intensities to jump between states: $\lambda = -\log(p)$, where p is the probability of staying in the same state and $1-p$ is the probability of switching between states (see Eq. 9 and 10). Thus, for the case of jumping from formal to informal state, we estimate $\lambda_2 = 0.2040$ implying a conditional probability $P_{22} = 81.55\%$. To calculate λ_1, we use the relationship between (λ_1, λ_2) and the informality size η, given by

$$\frac{\lambda_2}{\lambda_1+\lambda_2} = \eta \qquad \frac{\lambda_1}{\lambda_1+\lambda_2} = 1-\eta \tag{28}$$

These relationships are obtained by imposing that densities integrate to the stationary mass of individuals with respective agent types (informal and formal) as Achdou et al. (2022, eq. 32) do. Based on expression (28), we obtain $\lambda_1 = 0.816$ when $\eta = 0.2$, and $\lambda_1 = 0.1147$ when $\eta = 0.64$. These values imply a conditional probability P_{11} equals to 44.23% and 89.16%, respectively. This suggests that the probability that an informal agent stays in the same state the next period has been increased when η increases. Table 1 lists the values of the eight parameters.

Table 1

Parameter values

Parameter		Formal Agent	Informal Agent
Subjective discount rate	ρ	0.05	0.05
Borrowing limit	$\underline{a}$	30% $\times\min\{0.33, 1\}$	30% $\times\min\{0.33, 0.1\}$
Relative risk aversion	γ	0.15	0.3
Income level	y	1	0.33
Intensity to jump between states	λ	0.2040	0.816 ($\eta = 0.2$)
			0.1147 ($\eta = 0.64$)
Tax rate	τ	0.18	0
Interest rate premium	θ	0	0.02
Informal sector size	$\eta = \{0.2, 0.64\}$		

3.2 Numerical Solution

We then solve the PDEs system stated in subsection 2.3.2 using the *finite difference method* suggested by Achdou et al. (2022). This method discretizes the HJB and the Fokker-Planck equations by approximating the derivatives of the value function and the density of the distribution of the state variable. We also use the *implicit method* to solve the stationary system and the *Upwind scheme* to decide when to use the forward/backward approximation of the first derivative of the value function.

We start assuming a fixed interest rate. Then, agents take it as a given and solve their optimization problem (the HJB equation). Since we have a continuum of heterogeneous agents, a dynamic of the wealth distribution is necessary (the Fokker-Planck equation). Then, we evaluate if the initial interest rate clears the bond market. If that is not the case, a change in the interest rate is introduced, and then we start the process again.

4 Simulation

To evaluate the effects of informal workers' characteristics on the wealth and consumption distribution, we progressively add characteristics of the informal economy to the benchmark model until to obtain a fully characterized *informal economy*. For every new informality characteristic added to the model, we compare the statistics of wealth and consumption distribution of the informal economy with the benchmark one. This comparison sheds light on the distributional effects of informality in the whole economy and how the economic mechanism works with every characteristic.

We begin by increasing the informality size η from 0.2 (benchmark economy) to 0.64. Next, we consider that the informal agent does not pay taxes $\tau_1 = 0$. We plug this characteristic into our model with a high informality size ($\eta = 0.64$). Our third step is to introduce an interest rate premium paid by borrower-informal agents, who have $a < 0$. Finally, we consider preference heterogeneity between the formal and informal agents. Our informal economy has these four characteristics, and then we compare it with the benchmark economy.

4.1 What are the Effects of a Larger Informality Size (η)?

This section starts our analysis by adding a large informal sector to the benchmark economy. Our benchmark economy is the best proxy of a "formal" economy. It is characterized by an informality size of 20% ($\eta = 0.2$), which means that 20% of the population has a lower income than the remaining 80%. Furthermore, agents are identical in other characteristics, such as relative risk aversion, tax rate, and risk premium. What happens with the agents' optimal decisions when η increases? Moreover, what are the effects on wealth and consumption distribution when η increases? We answer these questions in the following paragraphs.

Policy functions. For our benchmark economy, the policy functions shown in Figure 2 suggest that the consumption level of the informal agent is always lower than the formal one for any wealth value.[4] This result is expected since the labor income of the informal agent is one-third of that of the formal agent. Second, the informal agent dissaves for any value of wealth. Does this mean that 100% of informal agents are borrowers? No, they are not, but a significant percentage of informal people are net borrowers (around 90% when $\eta = 0.2$). In contrast, the formal agent has positive savings for low levels of wealth. Interestingly, it is possible to have an informal and formal agent with the same wealth level in this economy, but the informal one dissaves while the formal one saves. Third, it is also possible to have informal and formal agents who are borrowers ($a < 0$) with different saving behavior.

But what happens when the informal size increases? The same Figure 2 shows the new policy functions when $\eta = 0.64$. Increasing η has two straightforward effects. First, the probability of staying in an informal state (P_{11}) increases significantly from 44.23% to 89.16% (see Eq. 28). Second, the economy is populated with more agents with low income (from $\eta = 20\%$ to $\eta = 64\%$).

With a higher P_{11}, informal agents increase their demand for bonds to self-insure against idiosyncratic shocks. This increase is reflected in a movement of savings upwards, indicating that informal agents are saving more (or dis-saving less). The consumption behavior of informal agents is influenced by two opposing effects through the marginal propensity to consume (MPC): while a lower equilibrium interest rate increases MPC, thereby increasing consumption, a higher P_{11} reduces MPC and

[4] A useful way to read policy functions is assuming that the agent enters the current period with wealth level a (horizon axis of graphs). Given this wealth level and the interest rate in equilibrium, the agent chooses optimally his consumption level. Then, as a result, saving is determined. Moreover, savings are interpreted as a change in wealth (da). For instance, consider an informal agent entering the current period with a wealth level $a = 0.1$, generating a saving equal to -0.15. His next period $t + \Delta$ wealth level is $a_{t+\Delta} = -0.05 = 0.1 + (-0.15) \equiv a_t + da$.

consequently, consumption.[5] Based on our calibration, the effect related to the probability of being in an informal state dominates over the effect of the interest rate; as a result, the policy function of consumption decreases.

Formal agents exhibit similar behavior: as idiosyncratic income risk increases, they engage in self-insurance by increasing savings and consequently reducing consumption. However, the impacts on consumption and saving are less pronounced compared to informal agents due to their higher incomes. It is worth noting that the interest rate plays a crucial role in the dynamics of consumption and saving, necessitating a careful explanation, which we provide in the following paragraph.

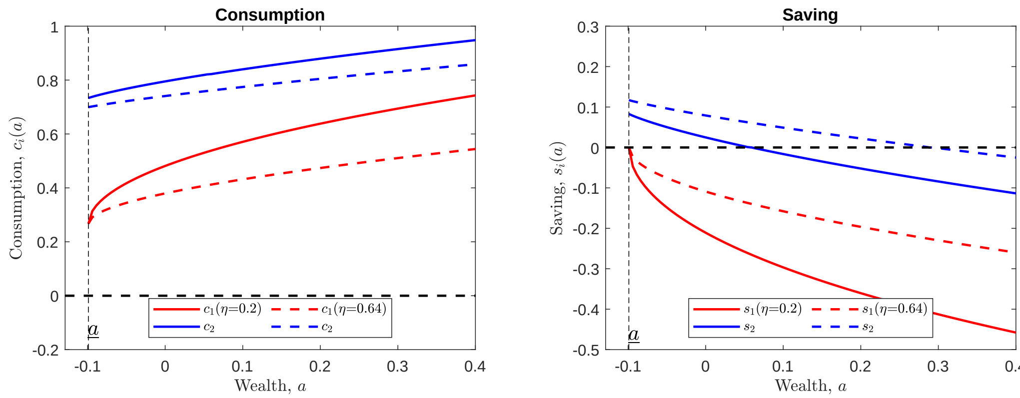

Figure 2
Policy Functions

Note. c_j and s_j represent consumption and savings of agent $j = \{1(\text{informal}), 2(\text{formal})\}$.

Interest rate. The equilibrium interest rate exhibits an inverted U-shape form with respect to η. The rationality is as follows: for values of η below 0.5, an increase in informality size pushes up the supply of riskless bonds since informal agents need funds for financing consumption given their low income. This behavior decreases the bond price and hence increases the interest rate. However, since the economy has a large formal sector $(1 - \eta > 0.5)$, which is willing to buy bonds from informal agents, the bond demand increases. This demand movement cancels out the initial reduction in bond price, resulting in a higher equilibrium bond price and, hence, a lower interest rate. In this case, the demand effect dominates the bond market's supply effect, resulting in a low equilibrium interest rate[6].

Nevertheless, for large informality size above $\eta = 0.5$, the dominance reverses: the supply effect dominates the demand effect. Informal agents liquidate their riskless assets, increasing the bond supply and hence pushing down bond prices. However, since the formal sector is smaller than the informal sector, it cannot absorb all the bond supply. Although the bond demand increases, its shift is not significant enough to cancel out the reduction in bond price. As a result, the equilibrium interest rate increases (see Figure 3).

[5]Following Achdou et al. (2022), the MPC for a CRRA utility function can be expressed as follows.

$$MPC_1 = f(\nu_1), \quad \nu_1 \approx (\rho - r)\frac{\underline{c}_1}{\gamma_1} + \lambda_1(\underline{c}_2 - \underline{c}_1), \tag{29}$$

where MPC is an increasing function of ν_1, which depends on r and λ_1 (related to P_{11}). Furthermore, $\underline{c}_1$ is defined as $\underline{c}_1 = (1 - \tau_1)y_1 + R\underline{a}$

[6]The bond supply side is formed by formal and informal agents with wealth below zero ($a < 0$), while the bond demand side is formed by formal and informal agents with wealth greater than or equal to zero ($a \geq 0$). The former set of agents are borrowers, and the latter are lenders

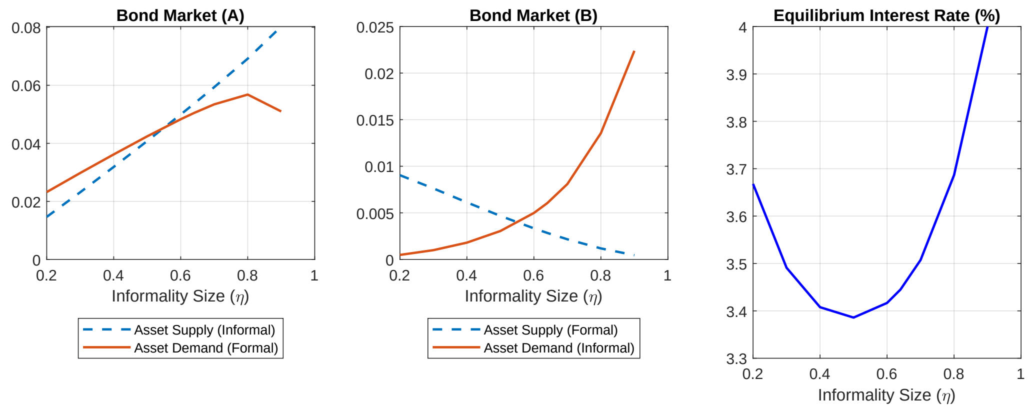

Figure 3
Asset Supply and Demand per Agent Type

Wealth and consumption distributions. We then examine the effects of larger η on wealth and consumption distribution. Figure 4 shows wealth and consumption densities for the benchmark economy ($\eta = 0.2$) and the informal economy ($\eta = 0.64$). At the aggregate level, increasing η moves the wealth and consumption distributions to the left side (see the left-hand graphs of Figure 4). It is also characterized by a higher concentration of the population on the borrowing constraint ($\underline{a}$) and at low consumption level. We then calculate the fraction of borrowers in the total wealth distribution, and we find that 41% are borrowers when $\eta = 0.2$ while 64% are borrowers when $\eta = 0.64$. This illustrates the concentration of wealth density below $a = 0$ when η increases.

A larger informality size also has significant effects on agent-type densities. The middle graphs of Figure 4 depict the effects of $\eta = 0.64$ for informal agents. First, there is a high concentration of informal agents around the borrowing limit ($\underline{a}$) compared to the benchmark economy. Our calculations reveal that the fraction of informal agents relative to the total population at the borrowing limit is 10% for $\eta = 0.2$, increasing significantly to 49% when $\eta = 0.64$. Second, their consumption levels align with the concentration around the borrowing limit, indicating that leveraged informal agents also exhibit lower consumption levels. A closer inspection of the distributions of formal agents (right-hand graphs of Figure 4) suggests that their wealth and consumption densities shift to the right, albeit with varying intensity: the movement of wealth density is more pronounced, whereas the change in consumption density is less significant.

Summary of effects. Table 2 shows two statistics of wealth and consumption distribution for the benchmark economy and informal economy (only with high η). We describe the effects at an aggregate level (total distribution) and agent-type level (distributions of informal and formal agents) as follows.

Aggregate level. The first takeaway is that a larger informality size significantly reduces the median of both wealth and consumption. For instance, for wealth, the median is not only lower but also its sign changes: from 0.018 for $\eta = 0.2$ to -0.094 for $\eta = 0.64$. This suggests that informality size shifts the wealth distribution to its left, implying the existence of more borrowers (from 41% of the population when $\eta = 0.2$ to 64% when $\eta = 0.64$). Furthermore, the same pattern appears for consumption but without changing its sign since the consumption can not be negative. The surprising result is its level: from approximately 0.8 to 0.3, representing a reduction of 60% when η increases from 0.2 to 0.64. This pattern is justified by the large fraction of borrowers in the economy, who have low consumption levels. That illustrates the significant effect of η in the economy. Another important

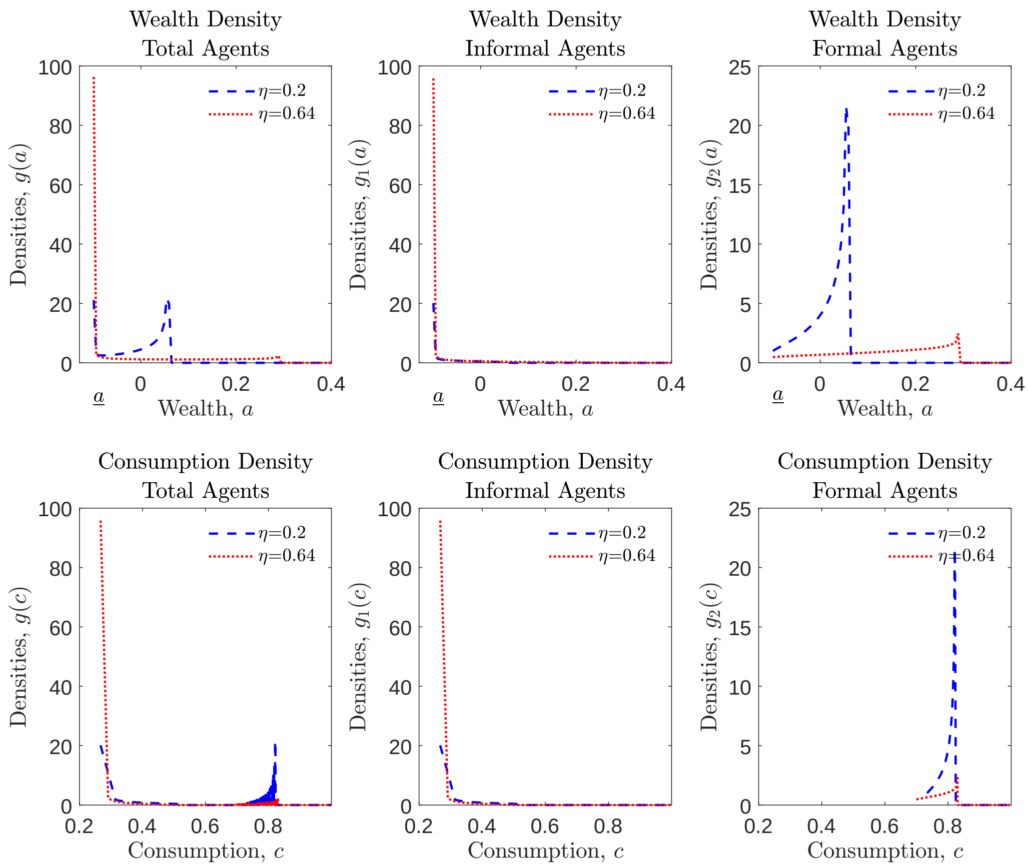

Figure 4
Wealth and Consumption Densities

result is the increase in the level of dispersion for both wealth and consumption. Quantitatively, the increment is economically significant–it is more than twice for wealth. This implies that economies with a higher informality sector create more inequality in wealth and consumption–a crucial characteristic of developing countries.

Agent-type level. An interesting result is that the median wealth of informal agents does not change when η increases. This is due to a *large* fraction of informal agents living at the borrowing limit ($\underline{a}$) with respect to the total informal-agent population. This ratio is equal to 51% (for $\eta = 0.2$), increasing to 76% (for $\eta = 0.64$). Since $\underline{a} = -0.099$ and more than 50% of the informal population have this level of wealth, we expect the median to be -0.099 for $\eta \in \{0.2, 0.64\}$. This contrasts with the median wealth of formal agents, which rises from 0.034 to 0.146.

Another result is that, although wealth dispersion increases for both agents, it is higher for formal agents. A different pattern is observed in the consumption distribution: while the dispersion decreases for informal agents, it increases for formal agents. What explains this result? One potential explanation is that more informal agents are living at or near the borrowing limit, leading to a similar consumption pattern. This slightly increases the median but decreases the dispersion. The higher dispersion in formal agents' consumption could be associated with a higher dispersion in their wealth.

Table 2
Distributional Effects of Informality Size

	η	Median Informal	Median Formal	Median Total	St Deviation Informal	St Deviation Formal	St Deviation Total	r (%)
Wealth Distribution								
Benchmark	0.2[a]	-0.099	0.034	0.018	0.041	0.042	0.055	3.7
High η	0.64	-0.099	0.146	-0.094	0.063	0.112	0.130	3.4
Consumption Distribution								
Benchmark	0.2[a]	0.2670	0.811	0.804	0.088	0.022	0.19	
High η	0.64	0.2672	0.789	0.291	0.052	0.036	0.24	

[a] Benchmark economy:

Low informality size, taxes paid by all agents, no interest rate premium, and similar risk aversion attitude

4.2 Adding No-tax Payment by Informal Agent ($\tau_1 = 0$)

In this section, we maintain the informality size (η) at 64% and introduce the second characteristic of an informal agent to our previous model: the absence of tax payment, $\tau_1 = 0$.

Policy functions. Figure 5 displays the new policy functions for both agent types. As we expect, the consumption policy function of the informal agent shifts up for every level of wealth. This movement is due to a no-tax payment denotes a positive income effect. Since this effect is permanent, the informal agent takes this into account in his optimal consumption, leaving a small effect on saving. The absence of tax payment also affects (but marginally) the formal agent's policy functions through the equilibrium interest rate. Specifically, the interest rate increases from 3.4% to 3.8%.

Given the high interest rate in equilibrium, it is useful to bring up the saving equation to gain insights into the policy functions.

$$s_j(a) = (1 - \tau_j)y_j + R_t a - c_j(a) \tag{30}$$

Borrower-formal agents ($j = 2$ with $a < 0$) now should pay higher interest for any wealth level below zero. Then, it is optimal for them to reduce their savings, increasing their consumption marginally. In contrast, lender-formal agents ($j = 2$ with $a > 0$) try to take advantage of the higher interest rate and increase their savings reducing their consumption. On the other hand, borrower-informal agents ($j = 1$ with $a < 0$ and $\tau_1 = 0$) balance the increase of the interest rate with the income effect such that the effect on their savings is insignificant. However, lender-informal agents experience two forces that strengthen each other: the income effect from no tax payments and a high interest rate. These effects encourage them to increase savings, especially for high levels of wealth.

The interest rate. The fact that the informal agent does not pay taxes increases the equilibrium interest rate. The mechanism is as follows: the income effect generated by the no-tax payment reduces the bond supply from borrower-informal agents ($a < 0$), pressing the price up and reducing the interest rate. However, this price effect is outperformed by the reduction of the bond demand from lending-informal agents, who also experience an income effect. In equilibrium, the bond price (interest rate) with $\tau_1 = 0$ is lower (higher) than with $\tau_1 = 18\%$ (see Figure 6).

Wealth and consumption distributions. We now examine the distributional effects of no-tax payment by informal agents (see Figure 7). At the aggregate level, $\tau_1 = 0$ has no effects on wealth distribution: the median and dispersion are (almost) the same as when $\tau_1 = 0.18$. However, this is not

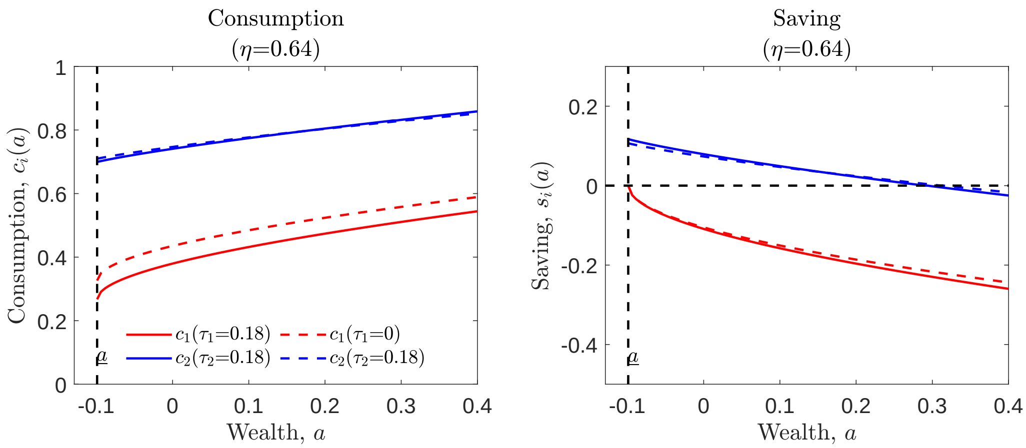

Figure 5

Policy Functions ($\tau_1 = 0$)

Note. c_j and s_j represent the consumption and saving of agent j, where $j = \{1(\text{informal}), 2(\text{formal})\}$.

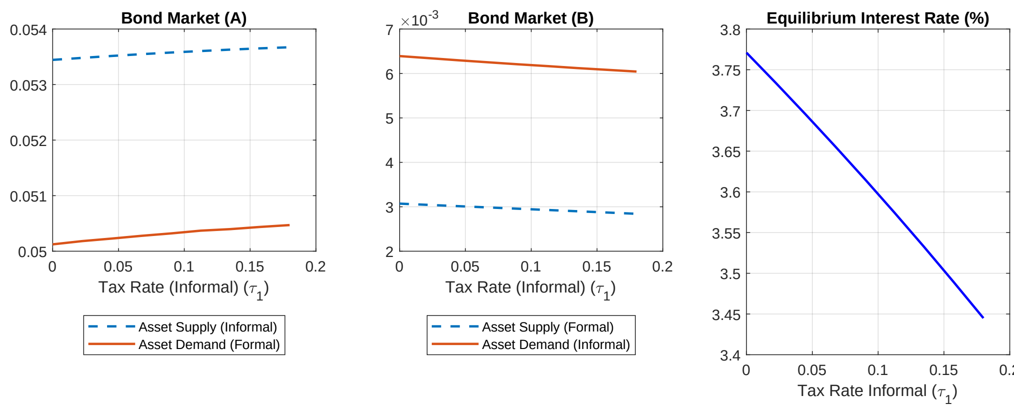

Figure 6

Bond Market and Interest Rate (τ_1)

the case for consumption distribution: the new economy has a higher median but lower dispersion. This is observed in the bottom-left graph of Figure 7, which shows a shift to the right of the total consumption distribution. To gain an understanding of these results, it is important to analyze what happens at the agent-type level. From the middle panel, it is clear that no-tax payment is beneficial for the informal population, shifting its consumption density to the right. In contrast, the effects on the consumption of formal agents are negligible. Since informal agents represent 64% of the total population, their consumption behavior is reflected in the aggregate consumption distribution.

Summary of effects. To observe the distributional effects of no-tax payment, we show two statistics (median and standard deviation) of the wealth and consumption distribution for the informal, formal, and total population in Table 3.

Aggregate level. The main effect of $\tau_1 = 0$ is on consumption distribution: its median increases by 23.4% (from 0.291 to 0.359), primarily due to the higher consumption of informal agents, while its dispersion decreases by 12.5% (from 0.24 to 0.21). Consequently, the absence of tax payments by

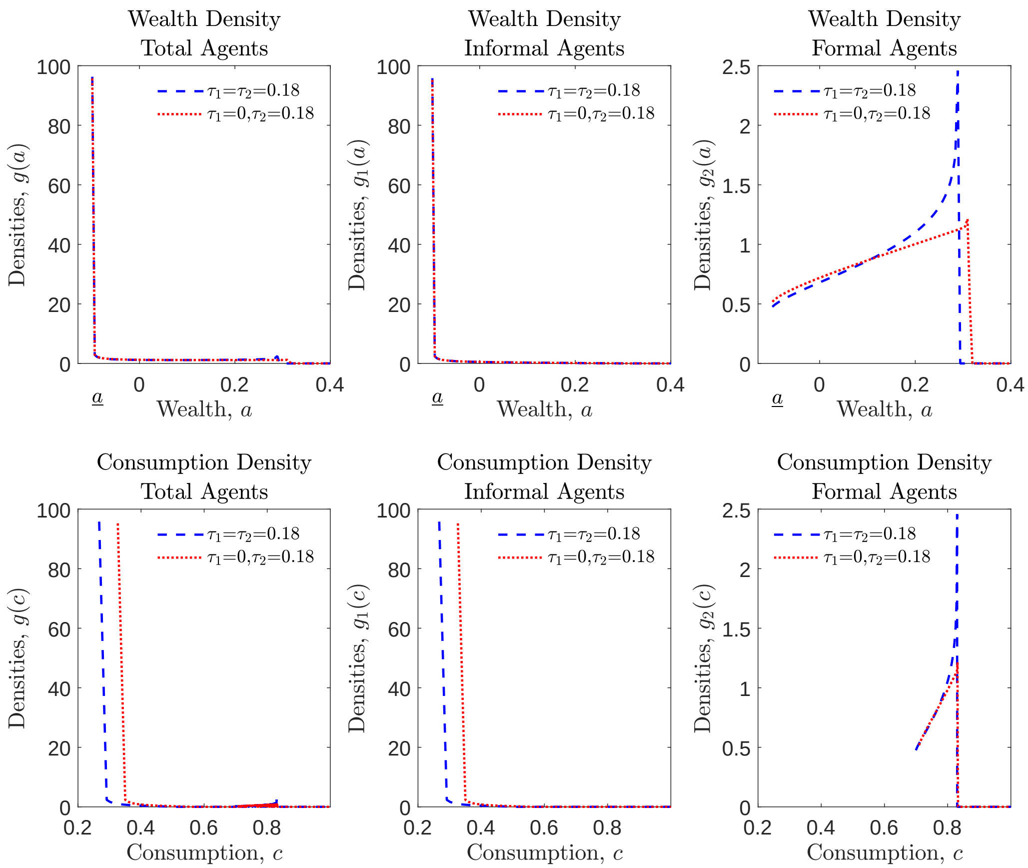

Figure 7
Wealth and Consumption Densities

informal agents causes the aggregate consumption distribution to shift to the right, albeit with low inequality. As for the wealth distribution, it is scarcely affected. While its median remains unchanged, its dispersion marginally increases.

Agent-type level. At this level, there are three main takeaways. First, the absence of tax payments by informal agents significantly increases their consumption median by 22% (from 0.2672 to 0.326) due to $\tau_1 = 0$ increasing the marginal propensity to consume of informal agents (see Eq. 29). Second, the consumption of formal agents is marginally affected, as their marginal propensity to consume remains relatively stable. Third, the consumption dispersion decreases for both agents: by 2% for informal agents and by 8% for formal agents. This illustrates the strength of the income effect caused by no-tax payment.

4.3 Adding Interest Rate Premium Paid by Informal Agent ($\theta_1 > 0$)

We then introduce the payment of an interest rate premium ($\theta_1 > 0$) to the model–the third characteristic of an informal agent. This premium is paid only by borrower-informal agents ($a < 0$), making the financing costly for them.

Policy functions. Figure 8 displays the policy functions of informal and formal agents. The first implication of the presence of a premium is that the informal agents' policy functions exhibit a kink

Table 3
Distributional Effects of No-tax Payment

	η		Median			St Deviation			r
			Informal	Formal	Total	Informal	Formal	Total	(%)
			Wealth Distribution						
Benchmark	0.2^{a}		-0.099	0.034	0.018	0.041	0.042	0.055	3.7
High η	0.64		-0.099	0.146	-0.094	0.063	0.112	0.130	3.4
+ No taxes	0.64	$\tau_I = 0$	-0.099	0.141	-0.094	0.065	0.117	0.131	3.8
			Consumption Distribution						
Benchmark	0.2^{a}		0.2670	0.811	0.804	0.088	0.022	0.19	
High η	0.64		0.2672	0.789	0.291	0.052	0.036	0.24	
+ No taxes	0.64	$\tau_I = 0$	0.3260	0.788	0.359	0.051	0.033	0.21	

[a] Benchmark economy:

Low informality size, taxes paid by all agents, no interest rate premium, and similar risk aversion attitude

in $a = 0$, clearly distinguishing the behavior of an informal borrower from an informal lender. Second, the informal agent optimally reduces the use of his wealth (negative saving but closer to zero) when he depends on external financing ($a < 0$). This implies an adjustment in his consumption, as we can see in his policy function. However, for high levels of wealth, the premium does not have an effect on the policy functions of the informal agent since he is a lender (or bondholder). Another takeaway from Figure 8 is that the premium seems to have effects on the consumption and saving of the formal agent. The natural question is how the premium affects the formal agent's policy functions. The answer necessarily involves the equilibrium interest rate. Since the borrower-informal agent should pay a premium, he reduces the bond supply pushing up the price and then reducing the interest rate (from 3.8% to 3.6%).

In this context, consider the borrower-formal agent ($a < 0$). This agent does not pay a premium θ. In fact, he faces a lower equilibrium interest rate for any debt level ($a < 0$) which means that the debt service is lower. This "extra inflow" can be used for consumption and saving. Given the labor income risk, the possibility of being informal in the next period, and hence obtaining a lower income, the borrower-formal agent tries to cover this risk by increasing his savings and marginally reducing his consumption. However, when his wealth stage changes ($a > 0$), the formal agent is now a lender (bondholder) changing his strategy: the lower interest rate discourages him from investing in bonds (saving). As a result, he allocates more funds to consumption, which reflects a negative relationship between consumption and interest rate.

Interest rate. We now explore the forces behind the equilibrium interest rate when the informal agent pays a premium for borrowing. The first expected effect is the reduction of bond supply from borrower-informal agents since external financing is costly. The initial reduction in the interest rate discourages lenders (formal and informal) from allocating funds to the investment opportunity, which reduces the bond demand. This reinforces the initial effect on the interest rate. As a result, the interest rate is lower in the new equilibrium (see Figure 9).

Wealth and consumption distributions. Figure 10 shows the median and standard deviation of wealth and consumption distribution for two economies. The first one is an informal economy with a large informality size ($\eta = 0.64$), no tax payment by the informal agent ($\tau_1 = 0$), and no interest rate premium $\theta_1 = 0$. This economy was explained in the previous section. The second economy

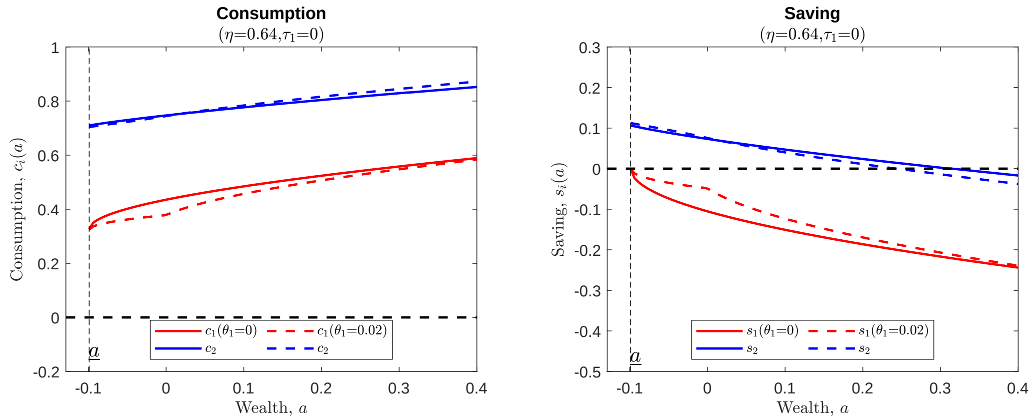

Figure 8

Policy Functions ($\theta_1 = 0.02$)

Note. c_j and s_j represent the consumption and saving of agent j, where $j = \{1(\text{informal}), 2(\text{formal})\}$.

keeps these characteristics except that now $\theta_1 = 0.02$–the borrower-informal agent pays a premium. Comparing both economies allows us to identify the effects of $\theta_1 = 0.02$.

The first effect at the aggregate level is that the concentration of the population in the borrowing constraint ($\underline{a}$) decreases, implying fewer people at the corresponding consumption level. Specifically, our calculation shows that the fraction of the population at the constraints decreases from 49% to 40%–illustrating the significant effect of interest premium. Second, two effects are observed on the consumption distribution: there is a shift to the left side, indicating that borrowing agents have lower consumption levels, while simultaneously, there is a shift to the right side, suggesting that lender agents increase their consumption.

When considering distributions by agent types, the initial observation is that the wealth distribution of informal agents is characterized by fewer individuals at the borrowing constraint and a higher proportion of unconstrained borrowers ($\underline{a} < a < 0$). Specifically, the fraction of informal agents relative to the total informal-agent population decreases from 76% to 61%, while the fraction of informal agents who are borrowers but are not at the constraint increases from 14% to 26%.[7] Taking both effects into account, the fraction of informal agents who are borrowers decreases from 90% to 87%. This indicates a rebalancing of the informal agent population due to the interest rate premium. Another interesting finding is that their consumption distribution shifts towards the left side, suggesting reduced consumption levels.

The effects differ for formal agents. Firstly, there is a migration of individuals from being borrowers ($a < 0$) to lenders ($a > 0$). Specifically, the fraction of formal agents who are borrowers decreases from 18% to 15%, while the fraction of those who are lenders increases from 82% to 85%. Although there is an effect on formal agents at the borrowing limit, it is small, shifting from 0.7% to 0.6%.[8]

Summary of effects. The last row of panel A and panel B of Table 4 show the effects of the premium on wealth and consumption distribution.

Aggregate level. There are three takeaways at the aggregate level. First, the standard deviation of wealth and consumption distributions is lower, with the most significant effect observed on wealth. Specifically, the standard deviation of wealth decreases by 11%, whereas it decreases by only 1% for

[7]These values are calculated considering the total population of informal agents.

[8]These values are calculated considering the total population of formal agents.

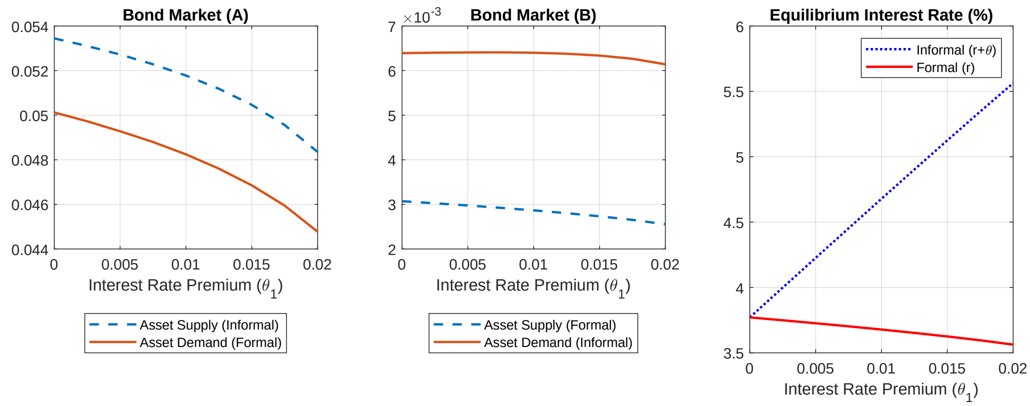

Figure 9

Bond Market and Interest Rate (θ_1)

Note. Asset supply (informal) is formed by borrower-informal agents ($a < 0$), and Asset supply (formal) is formed by borrower-formal agents ($a < 0$). On the other hand, Asset demand (informal) is formed by lender-informal agents ($a > 0$), and Asset demand (formal) is formed by lender-formal agents ($a < 0$).

consumption. Overall, this implies that the presence of a premium helps to reduce inequality. Second, the value of the median wealth distribution indicates that the economy is less leveraged by 38%, a significant effect. This is due to the external financing being now costly. Third, the median of total consumption increases by marginal by 1%; however, the main effects are observed within the agent-type distributions. Therefore, a premium has three aggregate effects: it reduces wealth inequality, reduces leverage, and has marginal effects on consumption.

Agent-type level. We proceed to analyze the informal and formal agent populations individually. Upon examining the wealth dispersion within each group, we note a reduction of 5% among informal agents and a more substantial 18% decrease among formal agents. This finding suggests that formal agents witness a more pronounced reduction in inequality in the presence of an interest rate premium. This result arises from borrowing-formal agents transitioning to become lenders due to the increased cost of external financing. Specifically, the proportion of lender-formal agents among the total formal agents rises from 82% to 85%, contributing to a decrease in wealth dispersion.

Another interesting result is that while the consumption dispersion reduces for informal agents by 22%, it increases for formal agents by 6%. The significant reduction of constrained informal agents in the new equilibrium makes their wealth distribution more concentrated, thereby reducing their consumption dispersion. Surprisingly, the presence of a premium increases the consumption dispersion within the formal agent population. This deserves a further explanation as follows.

The left-hand graph of Figure 8 provides insights into this finding. Mechanically, the new consumption policy function of the formal agent is more inelastic than the previous one. In the aggregate, we have borrower-formal agents with lower consumption and lender-formal agents with higher consumption than in an economy without a premium. Consequently, the consumption dispersion is higher. An economic explanation of that is provided by the optimal behavior of formal agents: given a low interest rate at the new equilibrium (3.6%), the current interest payment of borrower agents is alleviated, increasing their savings and then decreasing their consumption.

The opposite occurs when the formal agent is the lender. In this case, a low interest rate reduces its bond demand accordingly, reallocating funds from saving to consuming. As a result, the lower consumption by formal borrowers and higher consumption by formal lenders increase the consumption

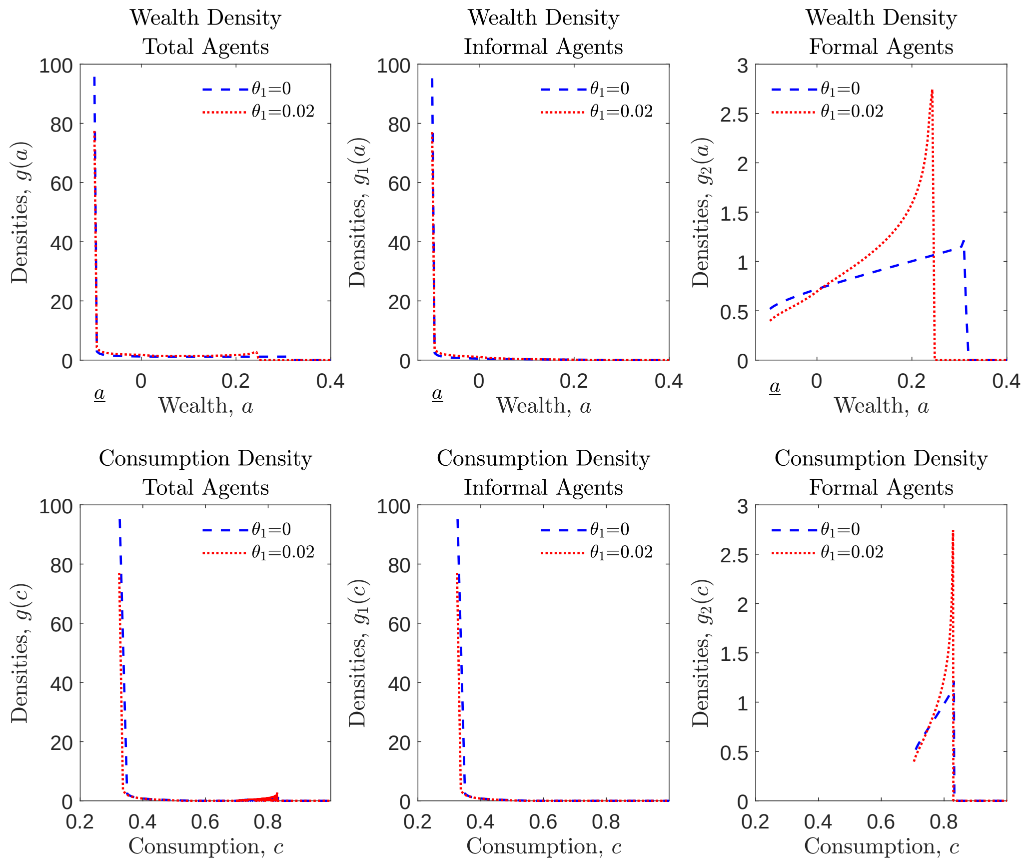

Figure 10
Wealth and Consumption Densities

dispersion in the formal population.

We then proceed to examine the impact on the median of consumption distribution for both agent types. Initially, we observe a 0.6% reduction for informal agents, contrasted with a 0.9% increase for formal agents. This disparity can be understood by considering policy functions. In the case of the formal population, the heightened consumption among lenders outweighs the reduced consumption among borrowers, owing to the smaller proportion of borrower-formal agents (15%) compared to lender-formal agents (85%). Consequently, the median consumption of formal agents primarily reflects that of lender-formal agents, who exhibit higher consumption levels.

However, this pattern does not hold for informal population. As shown in their consumption policy function (left-hand graph of Figure 8), consumption in the new equilibrium is lower across all wealth levels compared to the previous equilibrium. This result is due to two main factors: the risk of being informal in the next period and maintaining a low income, and, importantly, the risk of being a borrower, which implies paying an interest rate premium.

4.4 Adding Heterogeneity in Preferences ($\gamma_1 > \gamma_2$)

In this section, we study the last characteristic of an informal agent: a higher relative risk aversion parameter ($\gamma_1 > \gamma_2$). Specifically, we incorporate this characteristic into the previously studied economy. Consequently, we have a full characterized informal economy: with a large informality size ($\eta = 0.64$), no taxes paid by informal agents ($\tau_1 = 0$), borrower-informal agents subject to an interest

Table 4
Distributional Effects of Premium

	η		Median			St Deviation			r
			Informal	Formal	Total	Informal	Formal	Total	(%)
(A) Wealth Distribution									
Benchmark	0.2[a]		-0.099	0.034	0.018	0.041	0.042	0.055	3.7
High η	0.64		-0.099	0.146	-0.094	0.063	0.112	0.130	3.4
+ No taxes	0.64	$\tau_1 = 0$	-0.099	0.141	-0.094	0.065	0.117	0.131	3.8
+ Premium	0.64	$\theta_1 = 0.02$	-0.099	0.136	-0.058	0.062	0.096	0.116	3.6
(B) Consumption Distribution									
Benchmark	0.2[a]		0.2670	0.811	0.804	0.088	0.022	0.19	
High η	0.64		0.2672	0.789	0.291	0.052	0.036	0.24	
+ No taxes	0.64	$\tau_1 = 0$	0.3263	0.788	0.359	0.051	0.033	0.2125	
+ Premium	0.64	$\theta_1 = 0.02$	0.3245	0.795	0.362	0.040	0.035	0.2144	

[a] Benchmark economy:

Low informality size, taxes paid by all agents, no interest rate premium, and similar risk aversion attitude

rate premium ($\theta_1 = 0.02$), and informal agents exhibit greater risk aversion ($\gamma_1 = 0.3$).

Policy functions. Given that informal agents are now more risk averse ($\gamma_1 > \gamma_2$) and preferences are represented by a CRRA utility function, the elasticity of intertemporal substitution (EIS) for informal agents, defined as $1/\gamma_1$, is lower. A lower EIS leads to a smaller increase in current consumption relative to future consumption when interest rates decrease, as individuals are less willing to consume now rather than save for the future. However, the equilibrium interest rate decreases significantly, from 3.6% to 1.7%, encouraging the current consumption. This is exactly that we observe in the consumption policy function of informal agents (Figure 11). The lower EIS of the informal agents also explains the reduction in their saving.

How does a reduction of EIS among informal agents affect the policy function of formal agents? Similar to the previous analysis, the primary economic mechanism at play is the interest rate. Interestingly, we find that the effects are akin to those observed with the interest rate premium, but are stronger. The main difference lies in the magnitude of the reduction in the equilibrium interest rate. While the presence of an interest rate premium leads to a 20 basic points (bps) decrease in the interest rate (from 3.8% to 3.6%), the reduction resulting from the new EIS is much stronger–a 200 bps decline, pushing the interest rate down from 3.6% to 1.7%. With a lower interest rate, the borrower-formal agents increase their saving, reducing consumption. However, the effect is opposite for individual with high wealth levels.

Interest rate. Why is the interest rate lower in the new equilibrium? With low EIS, the informal agent prefers more current consumption, implying that the borrower-informal agents increase the bond supply, while lender-informal agents reduce their bond demand. Both effects increase the interest rate. This presents an opportunity for lender-formal agents (bond demand agents) in the economy, as they are the less risk-averse individuals. They increase their bond demand and consequently reduce the interest rate. This effect on the interest rate is reinforced by the borrower-formal agents, who reduce their bond supply given the high interest rate.

Therefore, we have two opposing forces on the interest rate: one that increases it (mainly from borrower-informal agents) and another that decreases it (mainly from lender-formal agents). The second force surpasses the first one due to formal agents being more willing to substitute consumption

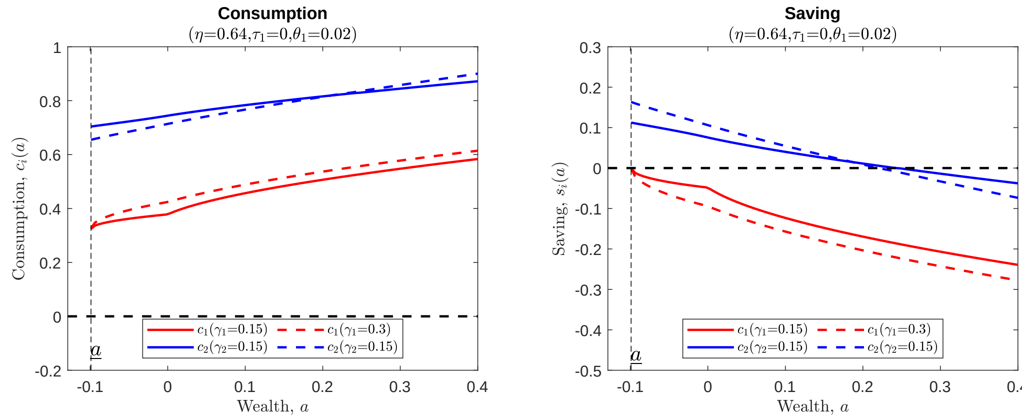

Figure 11

Policy Functions ($\gamma_1 = 0.3$)

Note. c_j and s_j represent the consumption and saving of agent j, where $j = \{1(\text{informal}), 2(\text{formal})\}$.

over time, thus increasing their savings by investing in bonds (see Figure 12). Our calculation shows that the bond supply of borrower-informal agents increases by 9.2%, outweighed by 10.4% increase in bond demand from lender-formal agents. As a result, the interest rate decreases in equilibrium.

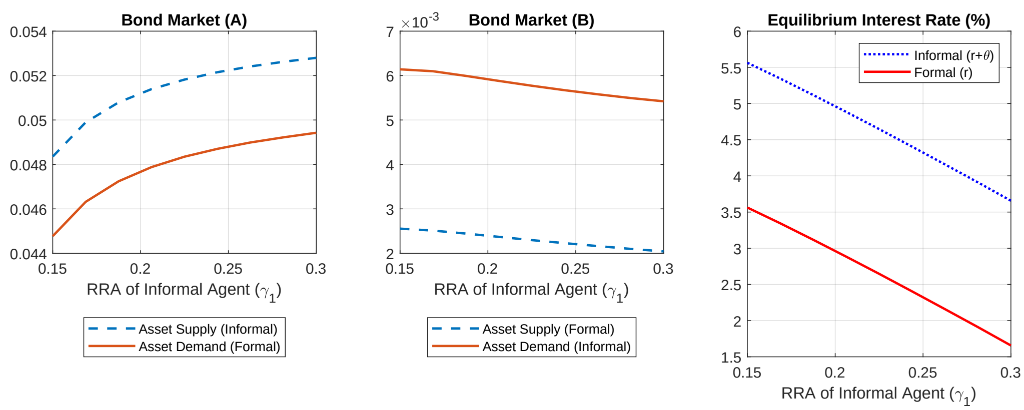

Figure 12

Bond Market and Interest Rate (γ_1)

Note. Asset supply (informal) is formed by borrower-informal agents ($a < 0$), and Asset supply (formal) is formed by borrower-formal agents ($a < 0$). On the other hand, Asset demand (informal) is formed by lender-informal agents ($a > 0$), and Asset demand (formal) is formed by lender-formal agents ($a < 0$).

Wealth and consumption distributions. We next explore the distribution effects shown in Figure 13. We identify four main patterns. First, at the aggregate level, there are more individuals at the borrowing constraint limit, increasing from 39% to 47%. Second, the consumption density shifts to the right, indicating an overall increase in consumption. Third, in the case of informal agents, there is a negligible effect on wealth distribution, while the main impact is observed in consumption. Fourth, concerning formal agents, their wealth distribution shifts to the right, but simultaneously, the density at lower levels of wealth increases.

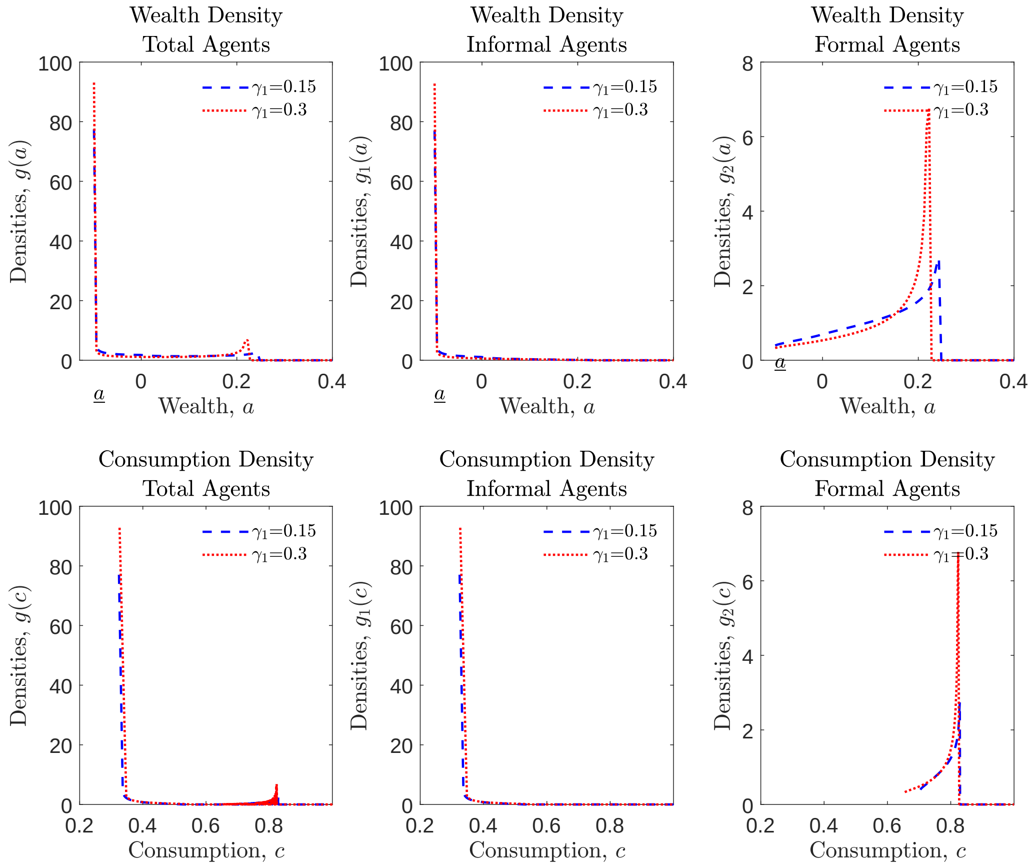

Figure 13
Wealth and Consumption Densities

Summary of effects. Table 5 shows the marginal effects of each informality characteristic on wealth and consumption distributions. We focus on the last row of Panel (A) and Panel (B), which displays the effects of a higher RRA of informal agents.

Aggregate level. With a higher RRA of informal agents, the economy becomes more leveraged in equilibrium (the median wealth increases by 58%) and exhibits greater inequality (the dispersion increases by 5.2%). In contrast, the median of aggregate consumption marginally increases by 0.3%, with a reduction in its dispersion by 1.3%.

Agent-type level. We now inspect the effects on agent types. First, there is no impact on the median of wealth of informal agents, but there is a strong effect on its dispersion (a reduction of 5%). The effects are much more significant for formal agents: their median wealth increases by 18%, while its dispersion reduces by 7%.

Second, the median and dispersion of the consumption distribution of informal agents increase by 0.6% and 22.5%, respectively, indicating that the main effect of a higher RRA (γ_1) is to generate more consumption inequality within this agent type. In the case of formal agents, the effect on consumption dispersion is much stronger, increasing by 31%.

Table 5

Distributional Effects of Preferences Heterogeneity

	η		Median Informal	Median Formal	Median Total	St Deviation Informal	St Deviation Formal	St Deviation Total	r (%)
(A) Wealth Distribution									
Benchmark	0.2^{a}		-0.099	0.034	0.018	0.041	0.042	0.055	3.7
High η	0.64		-0.099	0.146	-0.094	0.063	0.112	0.130	3.4
+ No taxes	0.64	$\tau_I = 0$	-0.099	0.141	-0.094	0.065	0.117	0.131	3.8
+ Premium	0.64	$\theta = 0.02$	-0.099	0.136	-0.058	0.062	0.096	0.116	3.6
+ $\neq$ RRA	0.64	$\gamma_I = 0.3$	-0.099	0.161	-0.088	0.059	0.089	0.122	1.7
(B) Consumption Distribution									
Benchmark	0.2^{a}		0.2670	0.811	0.804	0.088	0.022	0.19	
High η	0.64		0.2672	0.789	0.291	0.052	0.036	0.24	
+ No taxes	0.64	$\tau_1 = 0$	0.3263	0.788	0.359	0.051	0.033	0.2125	
+ Premium	0.64	$\theta = 0.02$	0.3245	0.795	0.362	0.040	0.035	0.2144	
+ $\neq$ RRA	0.64	$\gamma_1 = 0.3$	0.3264	0.796	0.363	0.049	0.046	0.2117	

[a] Benchmark economy:

Low informality size, taxes paid by all agents, no interest rate premium, and similar risk aversion attitude

5 The contribution of each characteristic of informality to wealth and consumption distributions

In the preceding sections, we examined the individual marginal effect tied to each of the four defining features of informality. Expanding on this analysis, the following section undertakes a comparison between the benchmark economy and the informal economy, in which all four aforementioned aspects of informality interact simultaneously. As in our earlier discussion, our primary focus continues to be the assessment of implications for wealth and consumption outcomes. In each case, we present the individual contribution of each of the four informality features modeled in this paper.

Table 6 shows our calculations. Specifically, the Marginal Effect column shows the percentage change between the *full* informal economy and the benchmark economy. To be precise, the *full* informal economy is characterized by: large informality size ($\eta = 0.64$), no tax paid by informal agents ($\tau_1 = 0$), interest rate premium by informal agents ($\theta_1 = 0.02$), and informal agents are more risk averse ($\gamma_1 = 0.3$). On the other hand, the benchmark economy features only a small degree of informality ($\eta = 0.2$), while keeping other characteristics (τ, θ, and γ) similar between informal and formal agents.

Wealth distribution. Based on our calculations presented in the Marginal Effect column of Table 6, we draw the following conclusions. First, the *full* informal economy exhibits lower median wealth and higher inequality than the benchmark economy. This suggests that informality has negative effects on the economy. Second, inequality also increases in each agent-type group, with a much stronger effect on formal agents. Third, the median wealth of formal agents increases significantly (almost 3.79 higher than those in the benchmark economy). However, informal agents remain highly leveraged. These findings underscore the significant effects of informality on both the overall wealth distribution and within each agent-type group.

Next, we investigate the characteristics of informal agents that influence our results. Specifically, we aim to identify the marginal contributions of η, τ_1, θ_1, and γ_1. We address this in the following paragraphs.

Aggregate level. Inspecting the first two rows of Table 6, we conclude the following. First, the large informality size ($\eta = 0.64$) and the higher RRA of informal agents ($\gamma_1 = 0.3$) drive the reduction of median total wealth. Second, these same characteristics explain the increase in dispersion. Additionally, the absence of tax payment from the informal agents appears to play a negligible role in median wealth and has only a marginal impact on its dispersion. Importantly, the presence of interest rate premium increases the median wealth and reduces its dispersion. These findings lead us to hypothesize a policy exercise as follows. A policy aimed at reducing the size of the informal sector would reduce wealth inequality and allow the economy to be less leveraged. Conversely, policies oriented toward incentivizing informal agents to pay taxes would have marginal effects on inequality and no impact on median wealth. Finally, policies aimed at reducing the financial frictions that informal agents face, as reflected in a higher interest rate premium, would not be suitable for wealth distribution purposes.

		Marginal Effect	Contribution (in p.p.)			
			$+\eta$	$-\tau$	$+\theta$	$\neq \gamma$
Total	Median	-583%	-610.5	0.0	194.2	-166.5
	Std. Dev.	123%	137.9	2.2	-27.2	10.3
Formal	Median	379%	333.1	-15.1	-15.1	75.7
	Std. Dev.	115%	168.7	11.9	-50.6	-14.7
Informal	Median	0%	0	0	0	0
	Std. Dev.	43%	52.4	4.4	-5.5	-8.0

Table 6
Marginal effects of informality on wealth distribution. The sum of the elements in each row is equal to the value in the Marginal Effect column. For instance, in the first row: -583% = -610.5 p.p + 0.0 p.p + 194.2 p.p -166.5 p.p. We calculate these values as follows. Let a be the median of total wealth, then its percentage change between the *full* informal and benchmark economy is given by

$$\frac{a_{rra} - a_{\eta_0}}{a_{\eta_0}} = \frac{a_{\eta_1} - a_{\eta_0}}{a_{\eta_0}} + \frac{a_{\tau_1} - a_{\eta_1}}{a_{\eta_0}} + \frac{a_{\theta_1} - a_{\tau_1}}{a_{\eta_0}} + \frac{a_{rra} - a_{\theta_1}}{a_{\eta_0}},$$

where $(a_{\eta_1} - a_{\eta_0})/a_{\eta_0}$ is the percentage change of median wealth between the informal economy with only a higher informality size ($\eta = 0.64$) and the benchmark economy ($\eta = 0.2$). Similarly, $(a_{\tau_1} - a_{\eta_1})/a_{\eta_0}$ represents the percentage change of median wealth between the informal economy characterized by a higher informality size and no tax paid by informal agents ($\eta = 0.64, \tau_1 = 0$) and the informal economy with only a higher informality size ($\eta = 0.64$). $(a_{\tau_1} - a_{\eta_1})/a_{\eta_0}$ and $(a_{\theta_1} - a_{\tau_1})/a_{\eta_0}$ are defined in the same manner. Calculations shown in this Table are based on results shown in Table 5.

Agent-type level. We next analyze the distribution of wealth of agent types (informal and formal). When considering informal agents, our initial finding is that the median wealth remains unchanged when considering any of the informality features. At first glance, this might appear surprising. However, it becomes clear when we take into account that the fraction of informal agents at the limit constraint relative to the total number of informal agents is close to 50% for the benchmark economy. This fraction is higher than 50% with the introduction of any characteristics of informal agents. For example, in an economy characterized by a high informality size, this fraction rises to 76%. Similarly, in the *full* informal economy, it stands at 74%. This suggests that the median wealth consistently aligns with the borrowing constraint level ($\underline{a} = -0.099$) across informality features.

A second finding is that the wealth dispersion among informal agents is primarily driven by higher informality size. Specifically, an increase in informality size ($\eta = 0.64$), along with the relatively minor impact of the absence of tax payment ($\tau_1 = 0$), increases wealth dispersion. Conversely, the other two features, interest rate premium and higher RRA, have the opposite effect, reducing wealth dispersion.

There are also important spillover effects on formal agents. First, the significant increase in the median and dispersion of wealth is mainly explained by a higher informality size. Second, the absence of tax payment by the informal agent and their financial constraints reflected in a higher interest rate premium (θ_1) reduces the median wealth of formal agents and has opposite effects on its dispersion. Since the external financing is now more costly because θ_1, the total demand for funds decreases. This affects the wealth of formal agents, as they are the main lenders in the economy. The absence of tax payments has the same effect: the income effect experienced by informal agents makes them less dependent on external funds, which affects the lenders. Furthermore, we observe that the higher RRA of informal agents has two different effects on the wealth of formal agents: it increases the median but reduces the dispersion.

Overall, these results demonstrate that the characteristics of informal agents have different effects, both in direction and magnitude, on the aggregate and agent-type wealth distributions. Therefore, policies aimed at reducing η or θ_1 have different impacts on the wealth distribution at the aggregate level and across agents.

Consumption distribution. We then examine the distributional effects of informality features at both the aggregate and agent-type levels. Our results are shown in Table 7, where the Marginal Effect column indicates the percentage change between the *full* informal economy and the benchmark economy. Based on this, we draw the following conclusions. First, a *full* informal economy is characterized by a lower median consumption but higher dispersion compared to the benchmark economy. Second, formal agents experience greater dispersion, indicating higher inequality in consumption, while informal agents exhibit less dispersed consumption values. Both dispersions are economically significant: the consumption standard deviation of formal agents is 1.1 times higher than that of the benchmark economy, whereas, for informal agents, it is 0.56. Third, the median consumption of formal agents decreases by 2%, while it increases by 22% for informal agents, compared to the benchmark economy.

We next extend our analysis to welfare. Since welfare can be calculated from the utility function, which monotonically depends on consumption, we can infer some effects on the distribution of welfare. First, informality, represented by four characteristics (η, τ_1, θ_1 and γ_1), reduces the median of aggregate welfare and creates more dispersion among agents. Furthermore, informal agents enjoy a greater welfare median at the expense of formal agents. Finally, informality exacerbates inequality among formal agents while reducing it among informal agents. These results suggest that informality has not only distributional effects on wealth and consumption but also on welfare. Therefore, the consideration of informality in economic policies appears to be of first-order importance.

Aggregate level. We next investigate the factors driving the lower median consumption and higher dispersion in the informal economy. Considering the first two rows of Table 7, we observe the following. First, the large size of informality reduces significantly the median of total consumption, while the other informality characteristics (such as no tax payment, interest rate premium, and high RRA) mitigate it, albeit to a lesser extent. Second, although the main driver of increasing dispersion is the size of informality, the absence of tax payment also plays an outstanding role in decreasing it. This suggests that the feature of non-paying taxes by informal agents helps reduce consumption inequality in the economy. The rationale behind this is that non-payment of taxes for informal agents represents an income effect, increasing their consumption and aligning it more closely with the level of formal agents. This effect is also evident in its impact on median aggregate consumption.

		Marginal Effect	Contribution (en p.p.)			
			$+\eta$	$-\tau$	$+\theta$	$\neq\gamma$
Total	Median	-55%	-63.8	8.5	0.4	0.1
	Std. Dev.	11%	26.3	-14.5	1.0	-1.4
Formal	Median	-2%	-2.8	-0.1	0.9	0.1
	Std. Dev.	110%	65.2	-11.5	5.1	50.7
Informal	Median	22%	0.1	22.1	-0.7	0.7
	Std. Dev.	-44%	-40.7	-1.3	-12.6	10.8

Table 7
Marginal effects of informality on consumption distribution. The sum of the elements in each row is equal to the value in the Marginal Effect column. For instance, in the first row: -55% = -63.8 p.p + 8.5 p.p + 0.4 p.p + 0.1 p.p. We calculate these values as follows. Let a be the median of total consumption, then its percentage change between the *full* informal and benchmark economy is given by

$$\frac{a_{rra} - a_{\eta_0}}{a_{\eta_0}} = \frac{a_{\eta_1} - a_{\eta_0}}{a_{\eta_0}} + \frac{a_{\tau_1} - a_{\eta_1}}{a_{\eta_0}} + \frac{a_{\theta_1} - a_{\tau_1}}{a_{\eta_0}} + \frac{a_{rra} - a_{\theta_1}}{a_{\eta_0}},$$

where $(a_{\eta_1} - a_{\eta_0})/a_{\eta_0}$ is the percentage change of median consumption between the informal economy with only a higher informality size ($\eta = 0.64$) and the benchmark economy ($\eta = 0.2$). Similarly, $(a_{\tau_1} - a_{\eta_1})/a_{\eta_0}$ represents the percentage change of median consumption between the informal economy characterized by a higher informality size and no tax paid by informal agents ($\eta = 0.64, \tau_1 = 0$) and the informal economy with only a higher informality size ($\eta = 0.64$). $(a_{\tau_1} - a_{\eta_1})/a_{\eta_0}$ and $(a_{\theta_1} - a_{\tau_1})/a_{\eta_0}$ are defined in the same manner. Calculations shown in this Table are based on results shown in Table 5.

Agent-type level. We next analyze the factors that explain the statistics of consumption distribution of agent types. Our first observation is that the non-payment of taxes by informal agents increases their median consumption, while the interest rate premium decreases it. Consequently, policies aimed at increasing tax payments by informal agents would reduce their consumption, whereas policies designed to reduce the interest rate premium would benefit these agents.

The second result is that the informality size (η) plays opposite roles in the consumption dispersion of informal and formal agents: η decreases dispersion for informal agents but increases it for formal agents. Lastly, it is noteworthy that a higher RRA of informal agents has a significant effect on the consumption dispersion of formal agents.

Overall, these findings indicate that the consumption distribution among agent types is influenced differently by informality characteristics.

6 Conclusions

In this paper, we develop a simplified heterogeneous-agent model to investigate how the characteristics of an informal economy affect consumption and wealth distributions at both the aggregate level and among agents. Our framework is built upon Achdou et al. (2022), with the addition of four characteristics of an informal economy observed in data: high informality levels, non-payment of taxes by informal agents, payment of an interest rate premium for external financing, and higher risk aversion among informal agents

Our approach involves introducing each characteristic into our benchmark economy and analyzing its effects on wealth and consumption distributions. By doing so, we can calculate the marginal effects of each feature, enabling us to identify their magnitude and direction. This process sheds light on our understanding of the distributional effects of informality—a crucial but often overlooked topic in developing economies.

Using this model, we demonstrate that a *full* informal economy is characterized by lower median total wealth and higher inequality, significantly impacting wealth distribution among agents. This economy also exhibits lower median consumption and higher consumption inequality. Additionally, we highlight that each informality characteristic has different effects, both in magnitude and direction, on wealth and consumption distributions. Therefore, informality, a critical feature of developing countries, should be considered of paramount importance for policy analysis.

While our goal is to develop a simplified framework that introduces common characteristics of the informal economy, we acknowledge some restricted assumptions. However, these limitations do not undermine the framework's potential for extension. For instance, future research could relax the assumption of equal borrowing limits for formal and informal agents, consider endogenous shifts between states influenced by policy interventions targeting informality reduction, incorporate financial frictions as in Fernández-Villaverde et al. (2023), or introduce nominal frictions to analyze the distributional effects of monetary policy in informal economies. We leave these extensions for future work.

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Appendix

A Uncertainty, Information Structure, and Beliefs

Uncertainty. The uncertainty in the economy is represented by a filtered probability space $\{\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P}\}$, in which a Poisson process q is defined. As in the Huggett (1993) model, the idiosyncratic income risk in our economy is represented by exogenous income shocks that follow a two-state Poisson process q. The agent with low income, along with other characteristics, is the *informal* agent, while the agent with high income is the *formal* one. We also assume that the true state of nature is completely determined by the sample paths of q on time.

Information structure. The σ-field $\mathcal{F}_t$ is interpreted as representing the information available at time t. Furthermore, $\{\mathcal{F}_t^q\}$ is the augmented filtration generated by q. Similar to models with heterogeneity in preferences, in our model, the complete information filtration is defined by $\mathbf{F} = \{\mathcal{F}_t^q\}$.

Beliefs. The probability measure $\mathcal{P}$ is interpreted as representing the agents' common beliefs. Furthermore, all the stochastic processes in our model are progressively measurable with respect to $\mathbf{F}$.

B Derivation of the Hamilton-Jacobi-Bellman and Fokker-Planck Equations

B.1 The Hamilton-Jacobi-Bellman Equation

The strategy: First, we start with the optimization problem in discrete time with length period Δ. Second, we take the limit of the HJB equation with respect to $\Delta \to 0$. This last step allows us to obtain the HJB equation in continuous time.

(a) Discrete-time (length period = Δ)

- Discount factor:
$$\beta(\Delta) = e^{-\rho\Delta} \tag{31}$$

- Income. Individual with income y_j in period t keeps their income in period $t+\Delta$ with probability $P_j(\Delta) = e^{-\lambda_j\Delta}$ and switch to state y_{-j} with probability $1 - P_j(\Delta)$

- The Bellman equation for this problem is:

$$V_j(a_t) = \max_{\{c_t\}} \left\{ U(c)\Delta + \beta(\Delta) \left[\underbrace{P_j(\Delta) V_j(a_{t+\Delta}) + (1 - P_j(\Delta)) V_{-j}(a_{t+\Delta})}_{E[V_j(a_{t+\Delta})]} \right] \right\} \tag{32}$$

s.t.

$$a_{t+\Delta} = \Delta((1-\tau_j)\, y_{jt} + R_j a_t - c_t) + a_t \tag{33}$$

$$a_{t+\Delta} \geq \underline{a} \tag{34}$$

For j =1,2.

Importantly, the expected value function of the agent j considering the possibility to have the same income y_j and to jump to the other income y_{-j} in the next period $t + \Delta$.

$$E[V_j(a_{t+\Delta})] = P_j(\Delta)V_j(a_{t+\Delta}) + (1 - P_j(\Delta))V_{-j}(a_{t+\Delta})$$

(b) Taking limit with $\Delta \to 0$.

$$\begin{aligned} \beta(\Delta) &= e^{-\rho\Delta} \Rightarrow \beta(\Delta) \approx 1 - \rho\Delta & (35) \\ P_j(\Delta) &= e^{-\lambda_j\Delta} \Rightarrow P_j(\Delta) \approx 1 - \lambda_j\Delta & (36) \end{aligned}$$

In the HJB equation (32)

$$\begin{aligned} V_j(a_t) &= \max_{\{c\}}\{U(c)\Delta + (1-\rho\Delta)\big[(1-\lambda_j\Delta)V_j(a_{t+\Delta}) + \lambda_j\Delta V_{-j}(a_{t+\Delta})\big]\} \\ V_j(a_t) - (1-\rho\Delta)V_j(a_t) &= \max_{\{c\}}\{U(c)\Delta + (1-\rho\Delta)\big[(1-\lambda_j\Delta)V_j(a_{t+\Delta}) + \lambda_j\Delta V_{-j}(a_{t+\Delta}) - V_j(a_t)\big]\} \\ \rho\Delta V_j(a_t) &= \max_{\{c\}}\{U(c)\Delta + (1-\rho\Delta)\big[V_j(a_{t+\Delta}) - V_j(a_t) \\ &\quad +\lambda_j\Delta[V_{-j}(a_{t+\Delta}) - V_j(a_{t+\Delta})]\big]\} \qquad (37) \end{aligned}$$

Dividing the Eq. (37) by Δ

$$\frac{\rho\Delta V_j(a_t)}{\Delta} = \max_{\{c\}}\left\{\frac{U(c)\Delta}{\Delta} + (1-\rho\Delta)\left[\frac{V_j(a_{t+\Delta}) - V_j(a_t)}{\Delta} + \lambda_j(V_{-j}(a_{t+\Delta}) - V_j(a_{t+\Delta}))\right]\right\}$$

Taking $\Delta \to 0$

$$\rho V_j(a_t) = \max_{\{c\}}\left\{U(c) + \lim_{\Delta\to 0}\left[\frac{V_j(a_{t+\Delta}) - V_j(a_t)}{\Delta}\right] + \lambda_j(V_{-j}(a_t) - V_j(a_t))\right\} \qquad (38)$$

It is worth noting that the value function depends on the state variable a and indirectly on the time. Then, the derivative of the value function is respect to its variable a and not respect to the time t. The term in "lim" in the Eq. (38) has a denominator Δ: a change in time. We need to change this denominator in change in a. With this goal in mind, we can use the expression for $a_{t+\Delta}$ from the Eq. (33) for:

$$\begin{aligned} \lim_{\Delta\to 0}\left[\frac{V_j(a_{t+\Delta}) - V_j(a_t)}{\Delta}\right] &= \lim_{\Delta\to 0}\left[\frac{V_j(a_t + \Delta((1-\tau_j)\,y_{jt} + R_j a_t - c_t)) - V_j(a_t)}{\Delta}\right] \\ &= \lim_{\Delta\to 0}\left[\frac{V_j(a_t + \Delta((1-\tau_j)\,y_{jt} + R_j a_t - c_t)) - V_j(a_t)}{\Delta((1-\tau_j)\,y_{jt} + R_j a_t + c_t)}\frac{\Delta((1-\tau_j)\,y_{jt} + R_j a_t + c_t)}{\Delta}\right] \\ &= \lim_{x\to 0}\left[\frac{V_j(a_t + x) - V_j(a_t)}{x}\frac{((1-\tau_j)\,y_{jt} + R_j a_t + c_t)}{1}\right] \\ &= V_j'(a_t)((1-\tau_j)\,y_{jt} + R_j a_t + c_t) \qquad (39) \end{aligned}$$

Considering Eq. (39) into Eq. (38), **the HJB** equation would be:

$$\rho V_j(a) = \max_{\{c\}}\{U(c) + V_j'(a)((1-\tau_j)\,y_j + R_j a - c) + \lambda_j(V_{-j}(a) - V_j(a))\} \qquad (40)$$

with the law of movement of the state variable (or the agent's saving) -Eq. (41)- and the borrowing constraint —Eq. (42).

$$
\begin{aligned}
a_{t+\Delta} &= \Delta((1-\tau_j)\,y_{jt} + R_j a_t - c_t) + a_t \\
\frac{a_{t+\Delta} - a_t}{\Delta} &= (1-\tau_j)\,y_{jt} + R_j a_t - c_t \\
\lim_{\Delta \to 0} \left[\frac{a_{t+\Delta} - a_t}{\Delta} \right] \equiv \dot{a}_t &= (1-\tau_j)\,y_{jt} + R_j a_t - c_t \qquad (41) \\
a_{t+\Delta} \geq \underline{a} &\rightarrow \lim_{\Delta \to 0} : a_t \geq \underline{a} \qquad (42)
\end{aligned}
$$

Remark:

- We know that the wealth $a \in [\underline{a}, \infty^{+}[$
- In the interior of the state space: $a_t > \underline{a}$
- For Δ arbitrary small: $a_t > \underline{a}$ implies $a_{t+\Delta} > \underline{a}$

$$
\lim_{\Delta \to 0} a_{t+\Delta} = a_t \text{ and we know } a_t > \underline{a} \qquad (43)
$$

So: If $a_t > \underline{a} \Rightarrow$ for Δ small $\Rightarrow a_{t+\Delta} > \underline{a}$

That implies the borrowing constraint $a_t > \underline{a}$ never binds in the interior of the state space.

B.2 The Fokker-Planck Equation

The next step is to derive a law of movement of the distribution of the state variable a. We follow the same strategy that we applied to the HJB equation. First, we start with a discrete-time approach assuming a length of period Δ small. Then, we enter to continuous-time approach taking the limit of Δ to zero.

1. Continuous time economy:
   - $\tilde{a}_t$: Wealth
   - $\tilde{y}_t$: Income, $y_t \in \{y_1, y_2\}$
   - $d\tilde{a}_t = s_j(\tilde{a}, t)dt$: $s_j(\tilde{a}, t)$ is the optimal saving policy function which comes from solving the HJB equation.
   - $g_j(a, t)$: The fraction of population with income y_j and wealth equals to a in period t. This reflects the state of the economy when $\tilde{a}_t = a$ (value).
2. Discrete time economy:

   $G_j(a, t)$ represents the fraction of population with income y_j and wealth below "a" in period t:

$$
G_j(a, t) = \text{Prob}(\tilde{a}_t \leq a, \tilde{y}_t = y_j) \qquad (44)
$$

   Evaluating $G_j(a, t)$ at the borrowing limit $\underline{a}$:

$$
G_1(\underline{a}, t) + G_2(\underline{a}, t) = 0, \quad \forall t \qquad (45)
$$

   Where:

- $G_1(\underline{a}, t)$ is the fraction of people with income y_1 and wealth lower or equals to "a"
- $G_2(\underline{a}, t)$ is The fraction of people with income y_2 and wealth lower or equals to "a"

Then,

$$G_1(\underline{a}, t) = G_2(\underline{a}, t) = 0 \tag{46}$$

$$\lim_{a\to\infty^+}[G_1(a,t) + G_2(a,t)] = 1, \tag{47}$$

where:

- $G_1(a,t)$: The function of people with income y_1, "$\lim_{a\to\infty} G_1(a,t)$"
- $G_2(a,t)$: The function of people with income y_2, "$\lim_{a\to\infty} G_2(a,t)$"

B.2.1 Law of motion for $G_j(a,t)$

We want to derive a *law of motion for G_j*; i.e., we are interested in *how G_j changes over time.* For instance, for agents type j, we want to calculate:

$$\frac{\partial G_j(a,t)}{\partial t} = \lim_{\Delta\to 0}\left[\frac{G_j(a,t+\Delta) - G_j(a,t)}{\Delta}\right] \tag{48}$$

To define the Eq. (48), we need to find an expression for $G_j(a, t+\Delta)$. Therefore, our goal is to find that expression. Since:

$$G_j(a,t+\Delta) = \text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j] \tag{49}$$

To find an expression of Eq. (49), we proceed in two steps. First, we compute $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$ without considering income change between t and $t+\Delta$. Second, we consider the change in income. We then use the $G_j(a, t+\Delta)$ obtained in the previous steps to calculate $\frac{\partial G_j(a,t)}{\partial t}$ —the law of motion for G_j.

Additionally, to find the law of motion for G_j, we need to answer the following question: "if a type j individual has wealth $a_{t+\Delta}$ at time $t+\Delta$, then what level of wealth $\tilde{a}_t$ did he have at period t?" We know the law of movement of the state variable a as a definition of saving:

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta S_j(\tilde{a}_t) \tag{50}$$

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta S_j(\tilde{a}_{t+\Delta}) \tag{51}$$

We use the equation (51) because it is convenient. Important: both equation -(50) and (51)- are the same, the difference is the former looks forward in time and the latter looks backward. Therefore, we will use Eq. (51):

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta S_j(\tilde{a}_{t+\Delta}) \tag{52}$$

Intuition: If $S_j(\tilde{a}_{t+\Delta}) < 0$ (this means the agent dissaves), his past wealth $\tilde{a}_t$ must have been larger than his current wealth $\tilde{a}_{t+\Delta}$.

Step 1. We analyze the wealth in $t+\Delta$ without considering the change in income. Fig. 14 illustrates the movement of the fraction of people from t to $t+\Delta$ that have wealth below a in $t+\Delta$. Specifically, under the assumption of dissaving, $s_j \leq 0$, Fig. 14 shows that the probability of wealth in $t+\Delta$ to be below than a (i.e., $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$) comes from two sources:

- First, the fraction of population X that already had wealth below a in t. Since they dissaves, their wealth in $t+\Delta$ would be lower than a.

$$X = \text{Prob}[\tilde{a}_t \leq a] \tag{53}$$

- Second, the fraction of population Y that had wealth higher than a in t, but since they dissaved in t, their wealth in $t+\Delta$ would be below than a for some threshold a_t^*.

$$Y = \text{Prob}[a \leq \tilde{a}_t \leq a_t^*] \tag{54}$$

We need to calculate the level of wealth in t that allows us to get a in $t+\Delta$: $\tilde{a}_{t+\Delta} = a$. Then, using the Eq. (52), we can obtain a_t^*:

$$\begin{aligned} \tilde{a}_t &= \tilde{a}_{t+\Delta} - \Delta S_j(\tilde{a}_{t+\Delta}) \\ a_t^* &= a - \Delta S_j(a) \end{aligned} \tag{55}$$

Introducing a_t^* from Eq. (55) into Eq. (54):

$$Y = \text{Prob}[a \leq \tilde{a}_t \leq a - \Delta S_j(a)] \tag{56}$$

With these two sources, we now can calculate $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$:

$$\begin{aligned} \text{Prob}[\tilde{a}_{t+\Delta} \leq a] &= X + Y \\ &= \text{Prob}[\tilde{a}_t \leq a] + \text{Prob}[a \leq \tilde{a}_t \leq a - \Delta S_j(a)] \\ \text{Prob}[\tilde{a}_{t+\Delta} \leq a] &= \text{Prob}[\tilde{a}_t \leq a - \Delta S_j(a)] \end{aligned} \tag{57}$$

Recall, this probability (Eq. 57) does not consider the change in income between t and $t+\Delta$. The next step is to consider the transition of income.

Step 2. Now, we consider the possibility that some people from t with income y_j and income y_{-j} could have income y_j in period $t+\Delta$. Fig. 15 illustrates the changes in income and their probabilities. In particular, to calculate the probability of the wealth in $t+\Delta$ to be below a given that the income in that period is y_1, we need to take into account two sources:

- First, the fraction of the population that had income y_1 in t and have the same income in $t+\Delta$. The probability to have the same y_1 income is $(1-\lambda_1\Delta)$:

$$Pr[y_{t+\Delta} = y_1 | y_t = y_1] = 1 - \lambda_1\Delta$$

- Second, the fraction of the population that had income y_2 and now, in $t+\Delta$, they have income y_1. The probability to pass from income y_2 in t to y_1 in $t+\Delta$ is $(\lambda_2\Delta)$:

$$Pr[y_{t+\Delta} = y_1 | y_t = y_2] = \lambda_2\Delta$$

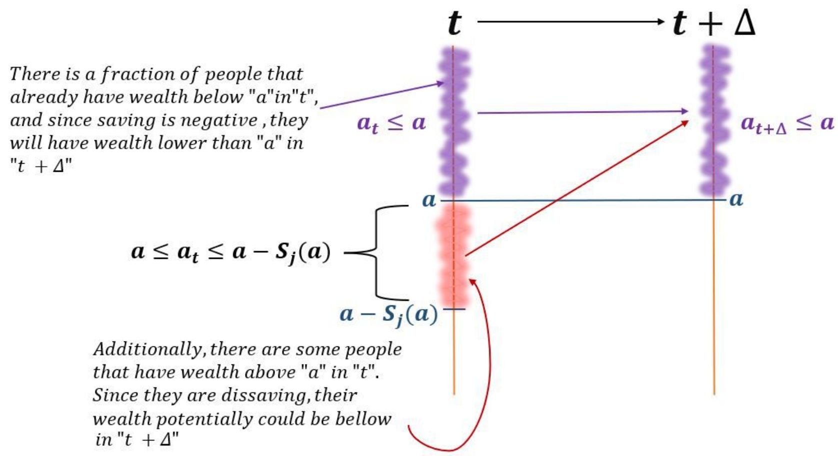

Figure 14
Transition of wealth

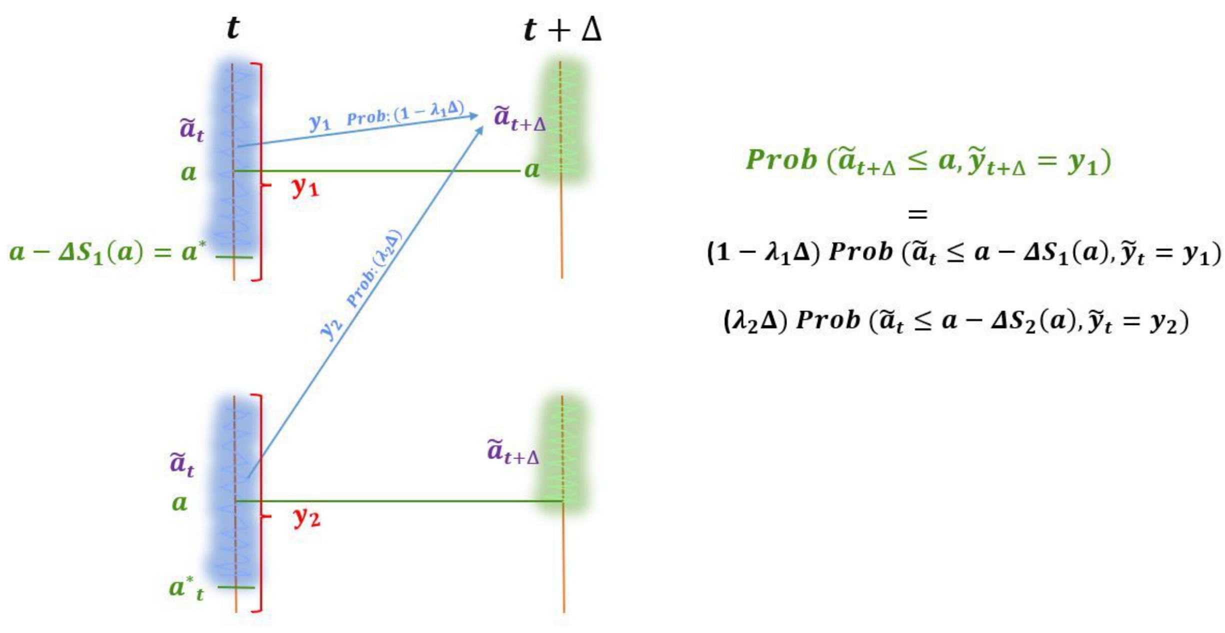

Figure 15
Transition of wealth with income

Taking into account these two sources, the probability of the wealth to be below a in $t+\Delta$ for income y_1 is:

$$\begin{aligned}
\text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_1] &= Prob[y_{t+\Delta} = y_1 | y_t = y_1]\text{Prob}[\tilde{a}_t \leq a - \Delta S_1(a), \tilde{y}_t = y_1] \\
&+ Prob[y_{t+\Delta} = y_1 | y_t = y_2]\text{Prob}[\tilde{a}_t \leq a - \Delta S_2(a), \tilde{y}_t = y_2] \\
\text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_1] &= (1-\lambda_1\Delta)G_1(a-\Delta S_1(a), t) + (\lambda_2\Delta)G_2(a-\Delta S_2(a), t)
\end{aligned}$$

In general terms, for any income y_j:

$$\underbrace{\text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j]}_{G_j(a,t+\Delta)} = (1-\lambda_j\Delta)G_j(a-\Delta S_j(a), t) + (\lambda_{-j}\Delta)G_{-j}(a-\Delta S_{-j}(a), t) \tag{58}$$

Calculating $\frac{\partial G_j(a,t)}{\partial t}$. So far, we have an expression for $G_j(a, t+\Delta)$. Next, we use this to calculate $\frac{\partial G_j(a,t)}{\partial t}$ —the law of motion for G_j. We start introducing $-G_j(a,t)$ in both side of Eq. (58):

$$\begin{aligned}
G_j(a, t+\Delta) - G_j(a,t) &= (1-\lambda_j\Delta)G_j(a-\Delta S_j(a), t) - G_j(a,t) + (\lambda_{-j}\Delta)G_{-j}(a-\Delta S_{-j}(a), t) \\
G_j(a, t+\Delta) - G_j(a,t) &= G_j(a-\Delta S_j(a), t) - G_j(a,t) \\
&- (\lambda_j\Delta)G_j(a-\Delta S_j(a), t) + (\lambda_{-j}\Delta)G_{-j}(a-\Delta S_{-j}(a), t)
\end{aligned} \tag{59}$$

Dividing Eq. (59) by Δ:

$$\begin{aligned}
\frac{G_j(a, t+\Delta) - G_j(a,t)}{\Delta} &= \frac{G_j(a-\Delta S_j(a), t) - G_j(a,t)}{\Delta} \\
&- \left[\frac{(\lambda_j\Delta)G_j(a-\Delta S_j(a), t) - (\lambda_{-j}\Delta)G_{-j}(a-\Delta S_{-j}(a), t)}{\Delta}\right]
\end{aligned} \tag{60}$$

Δ in the last term is ruled out from the numerator and denominator. As a result, Eq. (60) turns out:

$$\begin{aligned}
\frac{G_j(a, t+\Delta) - G_j(a,t)}{\Delta} &= \frac{G_j(a-\Delta S_j(a), t) - G_j(a,t)}{\Delta} \\
&- (\lambda_j)G_j(a-\Delta S_j(a), t) + (\lambda_{-j})G_{-j}(a-\Delta S_{-j}(a), t)
\end{aligned} \tag{61}$$

To make easy the algebra, we express the Eq. (61) in three terms:

$$A = B + C$$

where:

$$\begin{aligned}
A &= \frac{G_j(a, t+\Delta) - G_j(a,t)}{\Delta} \\
B &= \frac{G_j(a-\Delta S_j(a), t) - G_j(a,t)}{\Delta} \\
C &= -(\lambda_j)G_j(a-\Delta S_j(a), t) + (\lambda_{-j})G_{-j}(a-\Delta S_{-j}(a), t)
\end{aligned}$$

Next, we take $\lim_{\Delta \to 0}$ to the Eq. (61).

$$\lim_{\Delta \to 0} A = \lim_{\Delta \to 0} B + \lim_{\Delta \to 0} C$$

First term:

$$\lim_{\Delta \to 0} A = \lim_{\Delta \to 0} \left[\frac{G_j(a, t+\Delta) - G_j(a,t)}{\Delta} \right] = \partial_t G_j(a,t)$$

Second term:

$$\begin{aligned}
\lim_{\Delta \to 0} B &= \lim_{\Delta \to 0} \left[\frac{G_j(a - \Delta S_j(a), t) - G_j(a,t)}{\Delta} \right] \\
&= \lim_{\Delta \to 0} \left\{ \left[\frac{G_j(a - \Delta S_j(a), t) - G_j(a,t)}{-\Delta S_j(a)} \right] \left(\frac{-\Delta S_j(a)}{\Delta} \right) \right\} \\
&= \lim_{\Delta \to 0} \left\{ \left[\frac{G_j(a - \Delta S_j(a), t) - G_j(a,t)}{-\Delta S_j(a)} \right] (-S_j(a)) \right\} \\
&= \lim_{\Delta \to 0} \left\{ \left[\frac{G_j(a - \Delta S_j(a), t) - G_j(a,t)}{-\Delta S_j(a)} \right] \right\} (-S_j(a)) \\
&= \partial_a G_j(a,t)(-S_j(a))
\end{aligned}$$

Third term:

$$\begin{aligned}
\lim_{\Delta \to 0} C &= \lim_{\Delta \to 0} \left[-(\lambda_j) G_j(a - \Delta S_j(a), t) + (\lambda_{-j}) G_{-j}(a - \Delta S_{-j}(a), t) \right] \\
&= -(\lambda_j) \lim_{\Delta \to 0} [G_j(a - \Delta S_j(a), t)] + (\lambda_{-j}) \lim_{\Delta \to 0} [G_{-j}(a - \Delta S_{-j}(a), t)] \\
&= -(\lambda_j)[G_j(a,t)] + (\lambda_{-j})[G_{-j}(a,t)] \qquad (62)
\end{aligned}$$

Therefore, using the limit of A, B, and C, the Eq. (61) turns out:

$$\partial_t G_j(a,t) = -S_j(a)[\partial_a G_j(a,t)] - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t) \qquad (63)$$

We proceed with two more steps. First, we know a relationship between the CDF $G_j(a,t)$ and the density function $g_j(a,t)$:

$$\partial_a G_j(a,t) = g_j(a,t)$$

Using this relationship in the Eq. (63):

$$\partial_t G_j(a,t) = -S_j(a) g_j(a,t) - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t) \qquad (64)$$

Second, we derive the Eq. (64) with respect to "a":

$$\begin{aligned}
\partial_t \underbrace{\partial_a G_j(a,t)}_{g_j(a,t)} &= -\partial_a [S_j(a) g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t) \\
\partial_t g_j(a,t) &= -\partial_a [S_j(a) g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t) \qquad (65) \\
0 &= -\partial_a [S_j(a) g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a) \qquad (66)
\end{aligned}$$

The expression (65) is the Fokker-Planck equation over time for agent j. This partial differential equation captures the movement of the distribution $g_j(a,t)$ over time. To compute the stationary distribution, $g_j(a,t)$ keeps constant over time. We can obtain this stationary distribution considering $\partial_t g_j(a,t) = 0$, which is the Eq. (66).