BANCO CENTRAL DE RESERVA DEL PERÚ

Trend-Cycle Decomposition of GDP: A Flexible Filter

Fernando Pérez Forero*

* Banco Central de Reserva del Perú.

DT. N°. 2021-008
Serie de Documentos de Trabajo
Working Paper series
Diciembre 2021

Trend-Cycle Decomposition of GDP: A Flexible Filter *

Fernando J. Pérez Forero†

November 12, 2021

Abstract

The measurement of the Trend or long-term GDP is of vital importance for the characterization of macroeconomic scenarios. However, the usual filters are in some cases unstable when adding new data points and quite rigid in terms of their specification, which makes it difficult to calculate. This is especially relevant in contexts of greater uncertainty, where different shocks of varying magnitude can affect the aggregate economy. These filters are quite popular, although they could be unstable in very long series and with potential structural breaks. In this work a so-called 'flexible filter' is proposed, which is of the Cycle-Trend type, but which considers shocks with varying variances over time (Stochastic Volatility). The aforementioned filter is applied to quarterly data from the United States, Canada and Peru. In general, the consideration of a stochastic volatility component is a safe strategy against structural changes. Finally, the methodology also makes it possible to quantify the uncertainty associated with these estimates.

Resumen

La medición del PBI Tendencial o de largo plazo es de vital importancia para la caracterización de escenarios macroeconómicos. Sin embargo, los filtros usuales suelen ser en algunos casos inestables y bastante rígidos en cuanto a su especificación, lo cual dificulta el cálculo del mismo. Esto es en especial relevante en contextos de mayor incertidumbre, donde diferentes choques y de magnitud variable pueden afectar a la economía agregada. Dichos filtros son bastante populares, aunque podrían ser inestables en series muy largas y con potenciales quiebres estructurales. En este trabajo se propone un denominado 'filtro flexible', el cual es del tipo Ciclo-Tendencia, pero que considera choques con varianzas cambiantes en el tiempo (Volatilidad Estocástica). El filtro mencionado es aplicado a datos trimestrales de Estados Unidos, Canadá y Perú. En general, la consideración de un componente de volatilidad estocástica es una estrategia segura ante cambios estructurales. Finalmente, la metodología permite también cuantificar la incertidumbre asociada a dichas estimaciones.

JEL Classification: B41, C22, E32

Key words: Trend and Cycle Decomposition, State-Space Representation, Stochastic Volatility

*I would like to thank Fabio Canova, seminar participants at the BCRP and an anonymous referee for their helpful comments and suggestions. The views expressed are those of the author and do not necessarily reflect those of the Central Reserve Bank of Peru. All remaining errors are mine.

†Deputy Manager of Monetary Policy Design, Central Reserve Bank of Peru (BCRP), Jr. Santa Rosa 441, Lima 1, Perú; Email address: fernando.perez@bcrp.gob.pe

1 Introduction

The Trend-Cycle decomposition for time series has been widely used in the past for the characterization of macroeconomic facts with different patterns and properties (see e.g. Beveridge and Nelson (1981), Nelson and Plosser (1982), Harvey (1985), Clark (1987), Hodrick and Prescott (1997), Baxter and King (1999), Christiano and Fitzgerald (2003), among others). In addition, de-trending procedures used for time series transformation into stationary processes are necessary for the estimation of typical dynamic macroeconometric models such as Vector Autorregressions (VAR) through classical methods (in line with the typical Time Series textbooks, e.g. (Hamilton, 1994; Lütkepohl, 2005), among others) and Dynamic Stochastic General Equilibrium Models (DSGE) through bayesian methods (see e.g. Canova (2007), Herbst and Schorfheide (2016), among others.).

In this context, popular filters are typically used for this purpose such as the Hodrick-Prescott filter (Hodrick and Prescott, 1997) and the Band-Pass filter (Baxter and King, 1999; Christiano and Fitzgerald, 2003). In this family also enter the filters in the form of a dynamic state space system, i.e. as in Harvey (1985), Clark (1987) and extensions[1]. Canova (1998) puts discipline in this context performing a comparison of the cyclical components of macroeconomic time series using different de-trending methods, and states that the solely use of the Hodrick-Prescott filter might be problematic, since there exists a large variety of results across filters in terms of amplitude and duration of cycles[2]. In general, although they are extremely popular, these filters are subject to strong criticisms, since they could deliver unstable and distorted results in very long series and with potential structural breaks and outliers, a feature that complicates the task of time series de-trending. We mean with stability the fact that results can turn to be very different when adding few additional data points, sometimes changing dramatically the story of the previous estimates of unobserved components. This is especially relevant in contexts of high uncertainty, where different structural shocks with potentially time varying magnitudes

[1]See e.g. Harvey (1989) and Harvey and Trimbur (2003). In addition, Morley *et al.* (2003) demonstrates the equivalence between the Beveridge and Nelson (1981) decomposition and the Unobserved Component (UC) approach (Harvey, 1985).

[2]See also Cogley and Nason (1995) and more recently Hamilton (2018) and Hodrick (2020). See also the recent work of Canova (2020), who design a filter which reduces the biases of existing filters.

can affect the aggregate economy, as it was the case in the Covid-19 pandemic.

Regarding the issue of structural breaks, we have the seminal work of Hamilton (1989), and the extensions developed through bayesian methods by Kim and Nelson (1999). Furthermore, we have the papers of Perron and Yabu (2009), Luo and Startz (2014), Enders and Li (2015), Perron and Wada (2016), who find evidence of one or even multiple structural changes in macroeconomic time series. All these previous works consider a finite number of structural changes, which might be restrictive considering the multiple possibilities and the different nature of changes across time. For example, a regime change related with the Covid-19 pandemic should not be considered as similar as the Great Financial Crisis or even the Great Moderation.

For the reasons explained above, in this paper we propose a so-called 'flexible filter', which is based on the Trend-Cycle state-space specification, but also considers heteroskedasticity in the form of Stochastic Volatility[3]. In general terms, the consideration of a stochastic volatility component is a safe and flexible strategy against multiple structural changes (see Jacquier *et al.* (1994), Kim *et al.* (1998), Carriero *et al.* (2016), among others), and the contribution of this paper is to consider this extension, as a flexible specification, in order to correctly identify the time series components and the time varying measurement error. Simulation with Kalman filtering and smoothing techniques for these type of state space systems are usually related with the work of Carter and Kohn (1994) and Durbin and Koopman (2002), using the transformation of Kim *et al.* (1998) for considering stochastic volatility. We use the proposed filter for quarterly GDP of the United States, Canada and Peru.

Our results show that the gains of allowing for stochastic volatility are overwhelming. In first place, estimated trend, cycle components, as well as seasonal factors exhibit evidence of being smooth and well identified. In the particular case of the United States data, identified trend and cycle components are in line with the dates of the NBER recessions. In addition, we find evidence of significant time varying variances for each of the time series components, even for the seasonal factors both in raw data (non seasonally adjusted) and also seasonally adjusted ones. Evidence is symmetric for the three countries under consideration, and log-marginal likelihood

[3]See other trend-cycle models using heteroskedasticity in Perron and Wada (2016).

comparison validates the superiority of the model with time variances variances with respect to a restricted model with constant variances.

The document is organized as follows: section 2 describes the structural time series model, section 3 explains the estimation procedure, section 4 discusses the main results, and section 6 concludes.

2 A Structural Time Series model

Let y_t be the macroeconomic time series that will be the object of analysis (e.g. the real GDP in levels). Then, consider the following structural time series model, which can be applied for both quarterly ($S = 4$) and monthly data ($S = 12$), with $p < S$:

$$y_t = \mu_t + c_t + \gamma_t + \eta_t^y, \qquad \eta_t^y \sim N\left(0, \varepsilon_t^y\right) \tag{1}$$

$$\mu_t = \mu_{t-1} + b_{t-1} + \eta_t^\mu, \qquad \eta_t^\mu \sim N\left(0, \varepsilon_t^\mu\right) \tag{2}$$

$$b_t = b_{t-1} + \eta_t^b, \qquad \eta_t^b \sim N\left(0, \varepsilon_t^b\right) \tag{3}$$

$$c_t = \sum_{l=1}^{p} \phi_l c_{t-l} + \eta_t^c, \qquad \eta_t^c \sim N\left(0, \varepsilon_t^c\right) \tag{4}$$

$$\gamma_{t,1} = -\sum_{s=1}^{S-1} \gamma_{t-1,s} + \eta_t^\gamma, \qquad \eta_t^\gamma \sim N\left(0, \varepsilon_t^\gamma\right) \tag{5}$$

$$\gamma_{t,s} = \gamma_{t-1,s}, \quad s = 1, \ldots, S-1 \tag{6}$$

$$ln\left(\varepsilon_t^k\right) = ln\left(\varepsilon_{t-1}^k\right) + \epsilon_t^k, \qquad \epsilon_t^k \sim N\left(0, \sigma_k\right), \qquad k = (y, \mu, b, c, \gamma) \tag{7}$$

where μ_t is the trend component, c_t is the cyclical component, b_t is a time-varying intercept for the trend and γ_t is the vector of seasonal factors[4]. In addition, ε_t^k is the stochastic volatility component for each $k = (y, \mu, b, c, \gamma)$, which follows a log-random walk in order to ensure positive values for variances, and it is a simplified and more parsimonious specification of Kim *et al.*

[4]See details in Pelagatti (2016).

(1998) without an intercept and an autorregressive parameter, i.e. we have to estimate and identify a reduced number of parameters. It is important to clarify that all the stochastic innovations are treated as independent white noises, so that every factor or component is correctly identified. Last but not least, we consider in this specification the presence of a measurement error with potentially time varying variance. This is also an issue that should not be ignored, since there might be some special cases within the sample of analysis that are subject to this problem, such as financial crises episodes or even the Covid-19 pandemic.

The presented model is a generalization of the typical trend-cycle model (Clark, 1987), which belongs to the family of Unobserved Components (UC) models (Harvey, 1985, 1989). That is, the particular case without stochastic volatility, which implies that $\epsilon_t^k = 0$ and delivers as a consequence the case of constant variances $\varepsilon_t^k = \varepsilon_{t-1}^k = \varepsilon^k > 0$ for each $k = (y, \mu, b, c, \gamma)$, is the standard trend-cycle model with seasonal factors. In this context, the specification of popular filters such as Hodrick and Prescott (1997) can be achieved with further restrictions[5].

The model can be interpreted as additive or multiplicative, depending on the transformation used for the raw data. Typically, real GDP indicators are expressed in constant monetary units or as an index with a basis point in time. In both cases, if we introduce the observable in levels, i.e. $y_t = GDP_t$, the resulting model will be considered as "additive". On the other hand, if introduce the observable in logs, i.e. $y_t = 100 * ln\left(GDP_t\right)$ the resulting model will be considered as "multiplicative". The latter is the most common use for the application of different filters in the literature, so we will be aligned with this practice.

The model can be re-written as a state-space system with time varying matrices, so that:

$$y_t = D_t X_t + \varepsilon_t, \qquad \varepsilon_t \sim N\left(0, H_t\right) \tag{8}$$

$$X_t = A_t X_{t-1} + R_t \eta_t, \qquad \eta_t \sim N\left(0, Q_t\right) \tag{9}$$

Denote $\Theta = \left(\theta, X^T\right)$ as the parameter set of the model, with $X_t = (\mu_t, c_t, b_t, \gamma_t')'$ (see details in

[5]Harvey and Trimbur (2007) study the connection between the trend and cycle model in its state space form and the Hodrick-Prescott filter (see also Maravall and Del Río (2001) and Morley *et al.* (2003)).

Appendix A). Then, the complete posterior distribution for the parameter set Θ is:

$$p\left(\Theta \mid y^T\right) = p\left(\theta, X^T \mid y^T\right) \propto p\left(\theta\right) p\left(X_0\right) \prod_{t=1}^{T} p\left(y_t \mid X_t, \theta\right) p\left(X_t \mid X_{t-1}, \theta\right) \tag{10}$$

Notice that in this case the model is linear and normally distributed conditional on a value of the matrices H_t, A_t, Q_t, R_t. In line with that, the log-likelihood function evaluation and the analytical computation of the posterior distribution is feasible. In the next section we describe the main estimation algorithm.

In terms of the Kalman filter[6], the forecast distribution is

$$y_{t|t-1} \sim N\left(D_t X_{t|t-1}, D_t P_{t|t-1} D_t' + H_t\right)$$

and since y_t is scalar the probability density is equal to

$$\begin{aligned} f\left(y_{t|t-1}\right) &= (2\pi)^{-1/2} \det\left(D_t P_{t|t-1} D_t' + H_t\right)^{-1/2} \times \\ & \exp\left(\widetilde{y}_t' \left(D_t P_{t|t-1} D_t' + H_t\right)^{-1} \widetilde{y}_t\right) \end{aligned} \tag{11}$$

where $\widetilde{y}_t \equiv y_t - D_t X_{t|t-1}$ and $t = p+1, \ldots, T$.

The log-likelihood function $l\left(\theta \mid y^T\right) = ln\left(L\left(\theta \mid y^T\right)\right)$ of the system is given by:

$$l\left(\theta \mid y^T\right) = \sum_{t=p+1}^{T} \ln\left[f\left(y_{t|t-1}\right)\right] \tag{12}$$

which can be re-written as

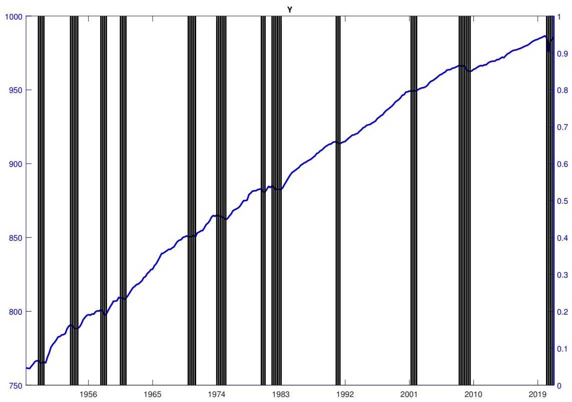
$$\begin{aligned} l\left(\theta \mid y^T\right) &= -\frac{(T-p)}{2} \ln\left[2\pi\right] - \frac{(T-p)}{2} \ln\left[\det\left(D_t P_{t|t-1} D_t' + H_t\right)\right] \\ & -\frac{1}{2} \sum_{t=p+1}^{T} \left(y_t - D_t X_{t|t-1}\right)' \left(D_t P_{t|t-1} D_t' + H_t\right)^{-1} \left(y_t - D_t X_{t|t-1}\right) \end{aligned} \tag{13}$$

[6] See e.g. Anderson and Moore (1979), Hamilton (1994) or Kim and Nelson (1999).

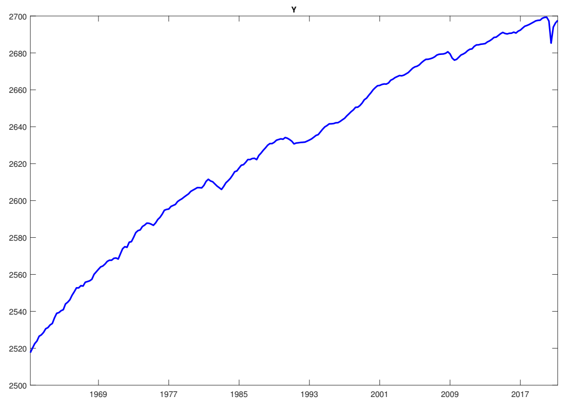
3 Bayesian Estimation

3.1 Data and estimation setup

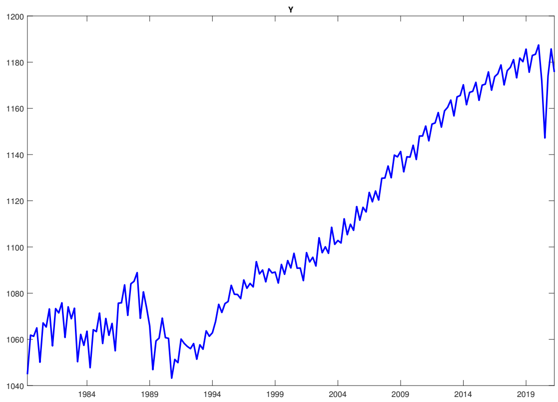
To illustrate the effects of the presented filter, we use quarterly real GDP data (as discussed in the previous section, we include data in logs and multiplied by 100) for the United States, Canada and Peru. Data from the first two countries is taken from the FRED Database, and the Peruvian data is taken from the Central Bank website. In the case of the United States, we also include the NBER recessions dates as shaded bars.



(a) United States (1947.Q1 - 2021.Q1)



(b) Canada (1961.Q1 - 2021.Q1)



(c) Peru (1979.Q1 - 2021.Q1)

Figure 1: Log GDP Data

It is important to mention that FRED data is seasonally adjusted, and the Peruvian data is not. Nevertheless, our presented filter is flexible relative to the previous transformations, so that we apply the same specification for each country. That is, even in the case of using seasonally adjusted data, where there is no unique method for that transformation, and they might deliver some problems with structural breaks, our setup is capable of capturing the remaining seasonal factors (and their volatility) from the data.

3.2 Priors

In the case of the stochastic volatility processes, we need to specify the distribution of the initial point $ln\left(\sigma_1^k\right)$ as follows:

$$ln\left(\varepsilon_1^k\right) \sim N\left(0, \nu_k\right) \tag{14}$$

with $k = (y, \mu, b, c, \gamma)$. Specifically, we set $\nu_y = \nu_\mu = \nu_c = \nu_b = \nu_\gamma = 1000$, i.e. we treat this as a diffuse filter since the prior law of motion of volatility is a random walk.

For the case of the variances σ_k, the distribution calibrates the *a priori* amount of time variation in the process:

$$\sigma_k \sim IG\left(d_{\sigma_k} \times \underline{\sigma}_k, d_{\sigma_k}\right) \tag{15}$$

where we set $d_{\sigma_y} = d_{\sigma_\mu} = d_{\sigma_c} = d_{\sigma_b} = d_{\sigma_\gamma} = 10$ and $\underline{\sigma}_y = \underline{\sigma}_\mu = \underline{\sigma}_c = \underline{\sigma}_b = \underline{\sigma}_\mu = 0.1$. A similar parametrization can be found in Carriero *et al.* (2016).

Then, the prior for the vector of coefficients ϕ is given by

$$\phi \sim N\left(\underline{\phi}, \underline{V}\right) \times I\left(\phi\right) \tag{16}$$

where $I\left(.\right)$ is the prior truncation for stationary draws, with $\underline{\phi} = \mathbf{0}_{dim(\phi)}$ and $\underline{V} = I_{dim(\phi)}$.

Finally, for the case of the state vector X_t, we specify the prior for the initial point as follows:

$$X_{p+1} \sim N\left(X_{0|0}, P_{0|0}\right) \tag{17}$$

In this case we set $X_{0|0} = 0_{dim(X)}$ and $P_{0|0} = 1000 \times I_{dim(X)}$ except for the stationary block associated with c_t, where we solve the Riccati's equation instead.

3.3 Gibbs Sampling

Given the priors and the likelihood function of the model, we proceed to estimate the set of parameters Θ. Using the Bayes theorem:

$$p(\Theta \mid Y) \propto p(Y \mid \Theta)\, p(\Theta) \tag{18}$$

Let $\Theta = \left\{\mu^T, b^T, c^T, \gamma^T, \varepsilon^{yT}, \varepsilon^{\mu T}, \varepsilon^{bT}, \varepsilon^{cT}, \phi, \sigma_y, \sigma_\mu, \sigma_b, \sigma_c, \sigma_\gamma, S_y^T, S_\mu^T, S_b^T, S_c^T, S_\gamma^T\right\}$ where S_i^T is the set of indicators associated with the mixture of normals approximation when simulating Stochastic Volatility (Kim *et al.*, 1998), and we put this block at the end of the simulation in line with the correction proposed by Del Negro and Primiceri (2015). We denote Θ/χ as the parameter vector Θ excluding χ.

We then set $l = 1$ and denote L as the total number of draws.

1. Draw $p\left(\varepsilon^{iT} \mid \Theta/\sigma^{iT}, Y^T\right)$, $k = (y, \mu, b, c, \gamma)$: Stochastic-Volatility
2. Draw $p\left(\phi \mid \Theta/\phi, y^T\right)$: OLS Regression
3. Draw $p\left(\sigma_i \mid \Theta/\sigma_i, y^T\right)$, $k = (y, \mu, b, c, \gamma)$: Inverse-Gamma
4. Draw $p\left(X^T \mid \Theta/X^T, y^T\right)$: State-Space
5. Draw $p\left(S_i^T \mid \Theta/S_i^T, y^T\right)$, $k = (y, \mu, b, c, \gamma)$: Discrete Distribution
6. If $l < L$, set $l = l + 1$ and go back to step 2.

We run the Gibbs sampler for $L = 100,000$ and discard the first $50,000$ draws in order to minimize the effect of initial values. In order to reduce the serial correlation across draws, we set a thinning factor of 50. As a result, we have $1,000$ draws for conducting inference. See details in Appendix B.

4 Results

4.1 Results for the United States

Results for the Postwar US GDP exhibit an estimated smooth trend component μ_t as it is depicted in Figure 2. The blue line is the median value, the red lines are the error bands for the credible set of 68% and the black line is the actual data.

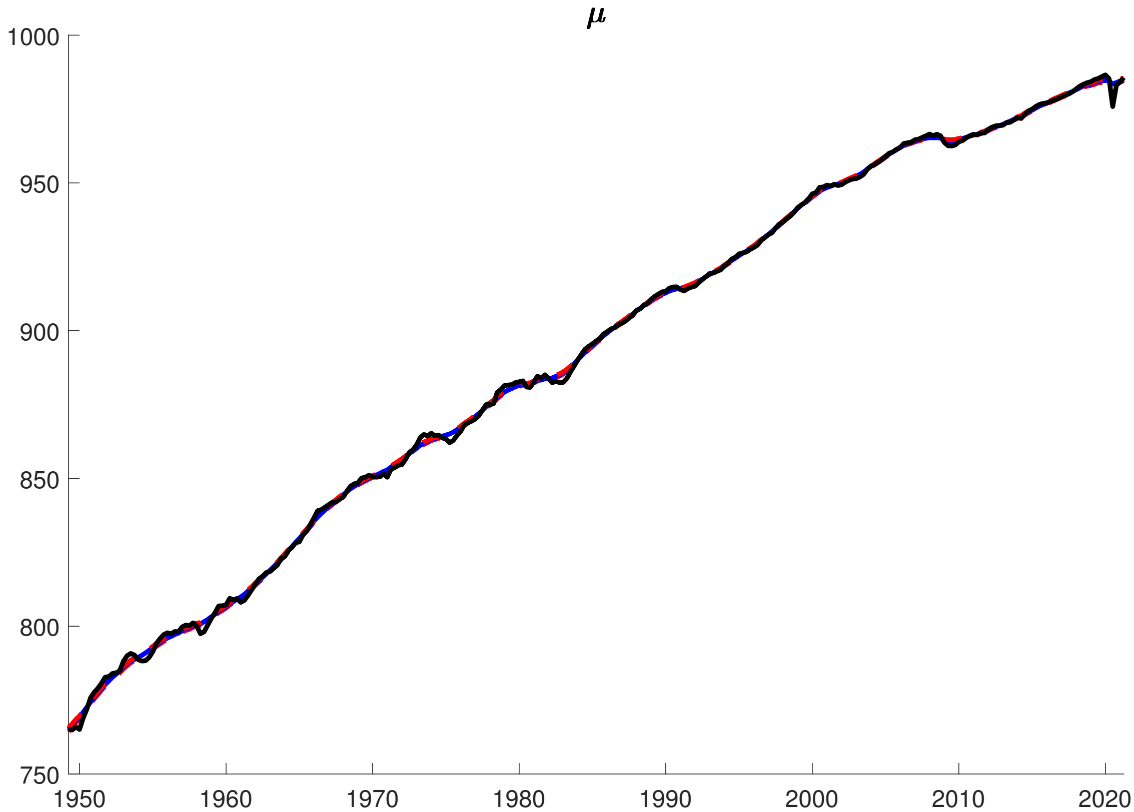
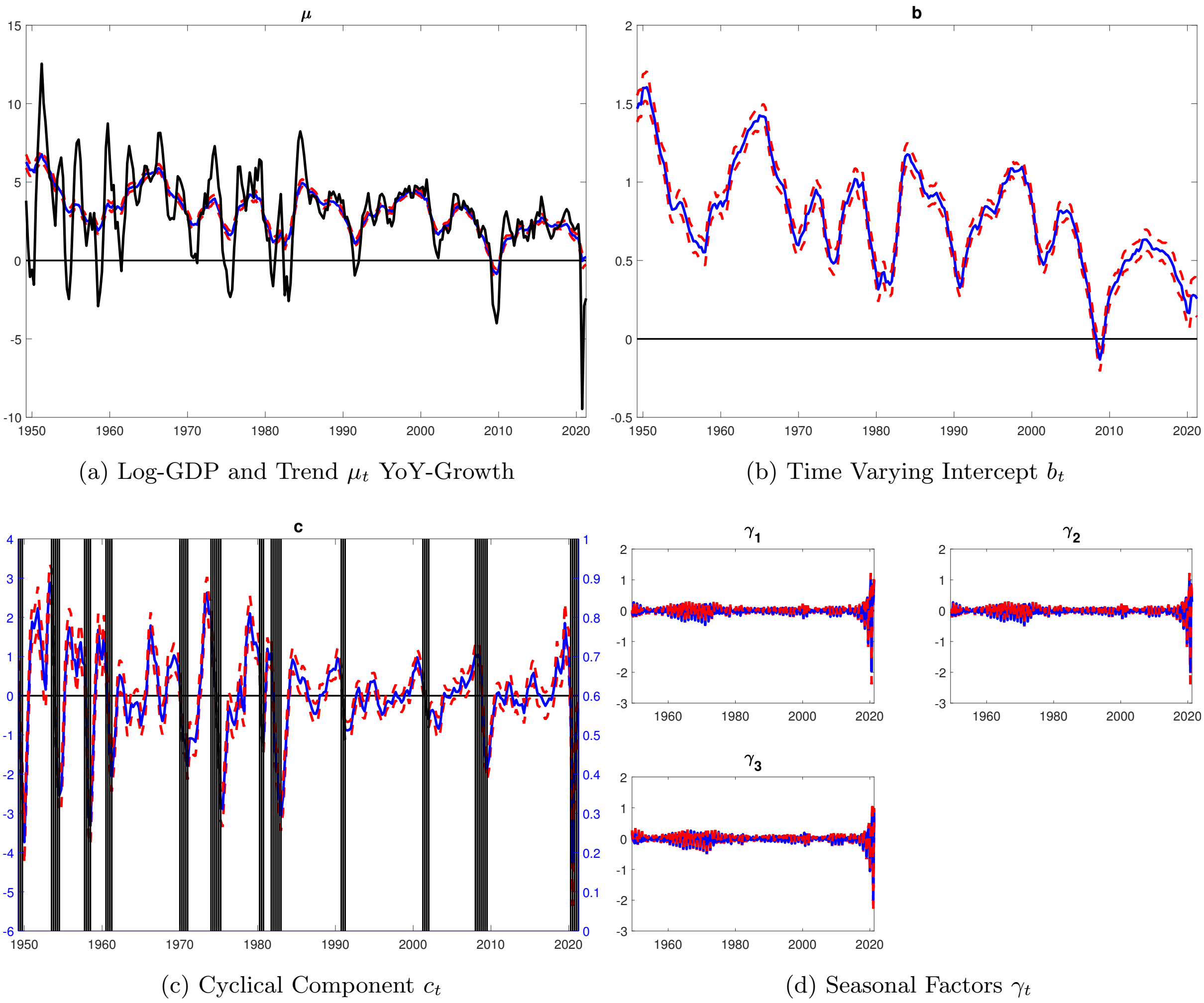


Figure 2: US: Log-GDP and Trend component μ_t (1947.Q1-2021.Q1)

In addition, the year-to-year growth rate of the estimated trend component μ_t is the compared with the GDP year-to-year growth rate (see panel (a) in Figure 3). As a result, trend growth rate is less volatile than GDP growth rate. This is a crucial point in order to correctly identify and disentangle permanent and transitory components within the context of structural breaks (besides the significant change in the time-varying intercept b_t depicted in panel (b). If we take a look to the Covid-19 pandemic episode, the model is capable to separate the contribution of permanent factors associated with μ_t from the change in GDP growth rate. In this regard, permanent factors could also be related with capital and labor utilization, as well as changes in total factor productivity[7]. Then, the rest of the change in growth can be attributed to the cyclical component c_t (see panel (c)), which coincides with the NBER recession dates for the full sample. Interestingly, even tough we have used seasonally adjusted data, so that the seasonal factors γ_t show a low magnitude for almost the full sample, this is not the case for the Covid-19 pandemic episode. That is, traditional seasonal-adjustment methods cannot deal with the Covid-19 shock, and therefore it is necessary to use a more flexible approach and also reinforces the idea of using raw data in order to correctly identify time series structural components.



(a) Log-GDP and Trend μ_t YoY-Growth

(b) Time Varying Intercept b_t

(c) Cyclical Component c_t

(d) Seasonal Factors γ_t

Figure 3: United States: Estimated GDP components

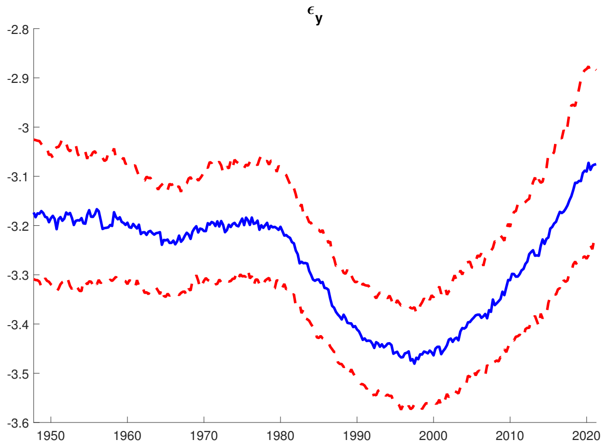
[7]See a large discussion about this theme in Hodrick (2020).

A result that is complementary to the previous one is the resulting estimated volatilities in Figure 4. In this case, time varying volatilities are consistent with the Great Moderation literature[8], where there is a significant decline in variances starting in the decade of 1980s, but these volatilities are also consistent with the end of the Great Moderation, starting with the Great Financial Crisis and exacerbated with the Tapering Tantrum and the recent Covid-19 pandemic episode. That is, time varying volatilities exhibit a significant change for the trend, cycle, seasonal and even the measurement error variance, showing statistical evidence in favor of the presented flexible approach compared to standard constant variances state space-system.

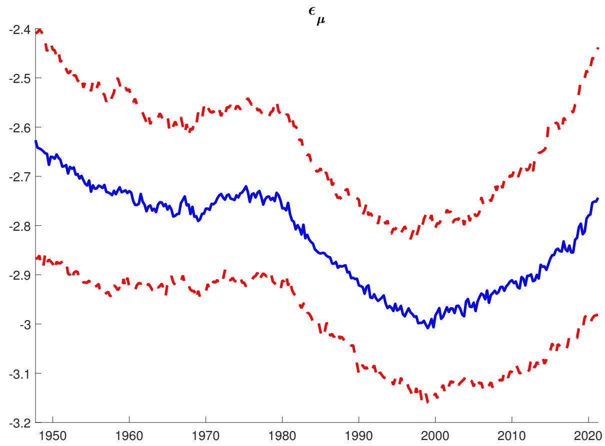
The previous result that shows statistical evidence of significant changes in variances could also be interpreted as the problem model misspecification. Of course, the misspecified model is the with constant variances (i.e. the standard trend-cycle model and the particular cases such as Hodrick-Prescott, Baxter-King, etc.), and here we validate our approach through the argument that using stochastic volatility is a safe strategy against the typical *omitted variable bias*. As a matter of fact, continuous changes in variances shed light on the idea that something important is missing, so that it reinforces the necessity of using richer models with a larger information set, such as multivariate ones.

Another valid interpretation of these results is the fact that the model should not be linear, and even that innovations should not be normally distributed, and that this is the main weakness of standard approaches. Thus, the more flexible is the approach, the more likely the results are reliable and truthful. Of course, the presented model can be extended in different directions, but the main conclusion should not change at all.

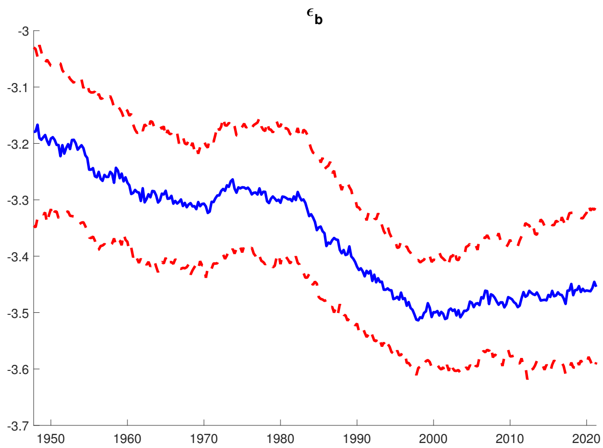
[8] See e.g. Stock and Watson (2003), Bernanke (2004), among others.



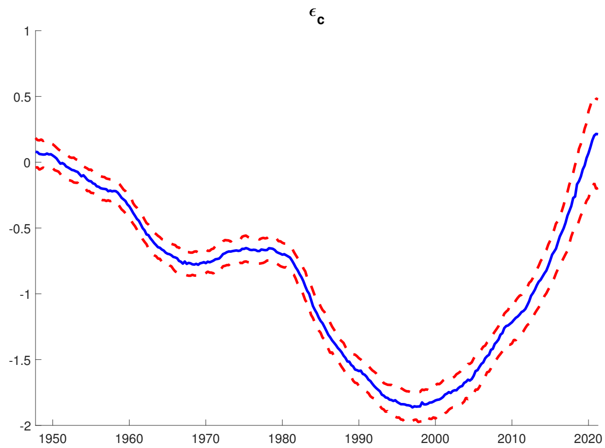
(a) Measurement Error



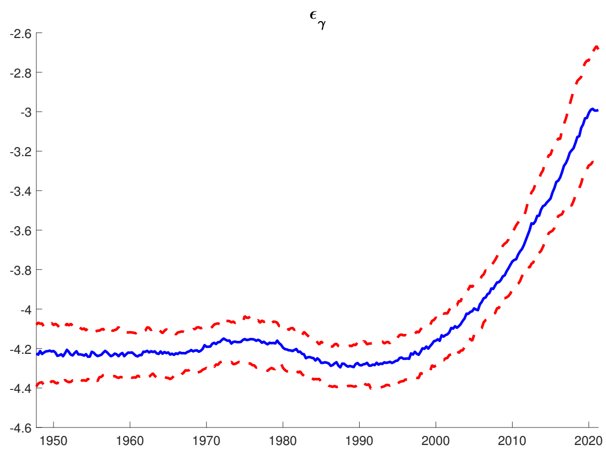
(b) Trend Error



(c) Intercept Error



(d) Cyclical Error c_t



(e) Seasonal Error γ_t

Figure 4: United States: Estimated Log-Volatilities

4.2 Results for Canada

We now turn to examine the results for Canada, where we also use seasonally adjusted data. In a similar fashion to the results obtained using US data, we observe in Figure 5 an estimated smooth trend component μ_t, which is particularly notorious for the two last episodes with large shocks, i.e. the Great Financial Crisis and the Covid-19 pandemic, as well as for the Great Moderation in early 1980s.

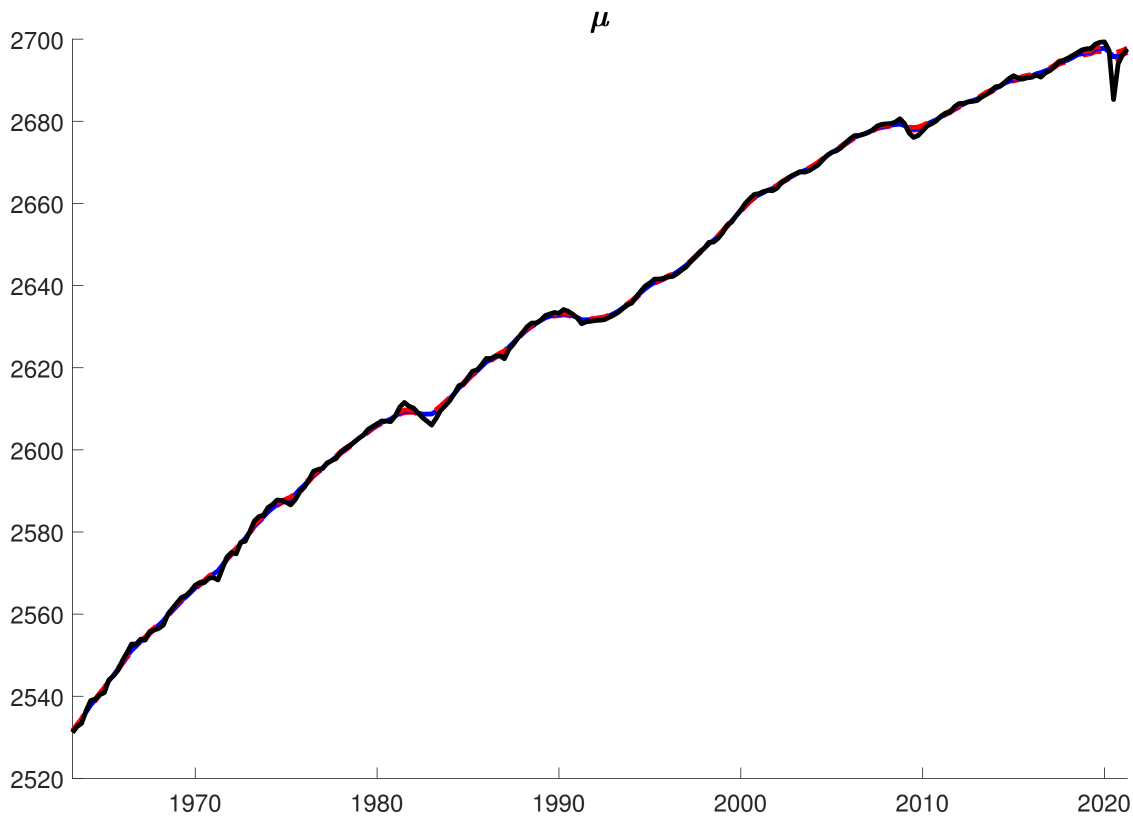
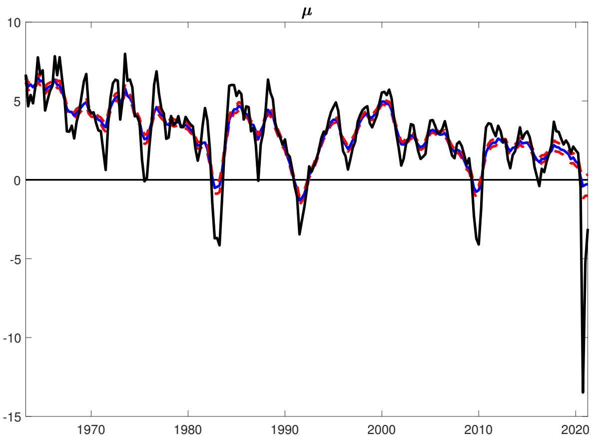
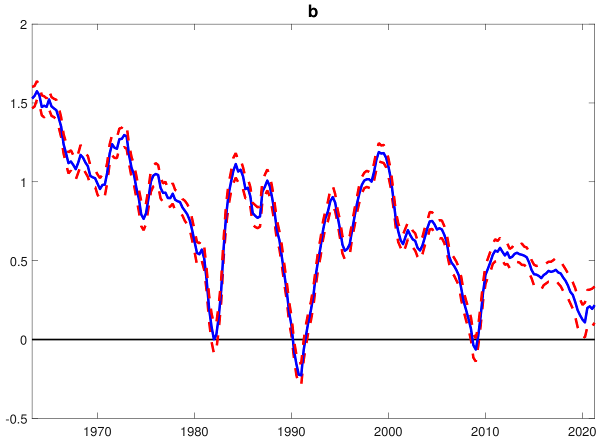


Figure 5: Canada: Log-GDP and Trend component μ_t (1961.Q1-2021.Q1)

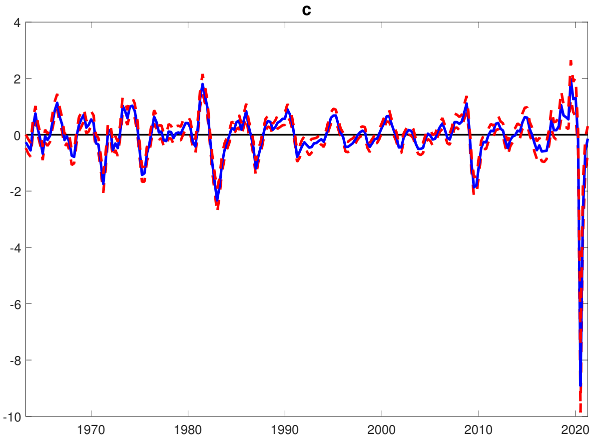
In all of the mentioned cases, the change in trend component μ_t is small relative to the actual change in GDP, as it is also depicted in Figure 6 (see panel (a)), i.e. the model captures a stable and smooth trend, and disentangles it from the cyclical component (panel (c)) and the seasonal factors (panel (d)). Notice that in this case we also find a significant and volatile vector of seasonal factors, although we are working with seasonally adjusted data.



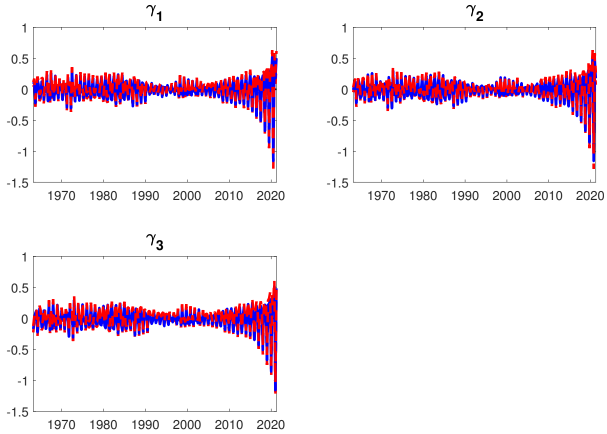
(a) Log-GDP and Trend μ_t YoY-Growth



(b) Time Varying Intercept b_t



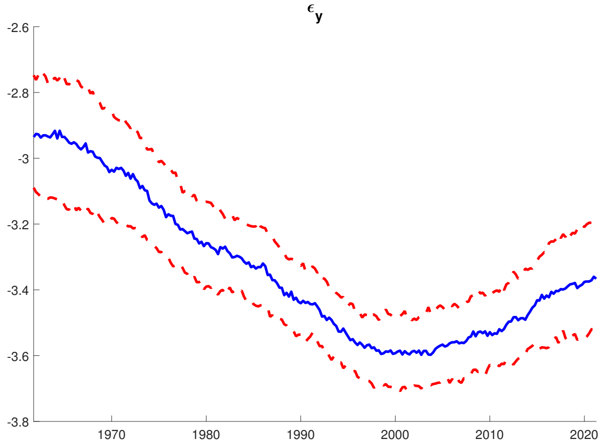
(c) Cyclical Component c_t



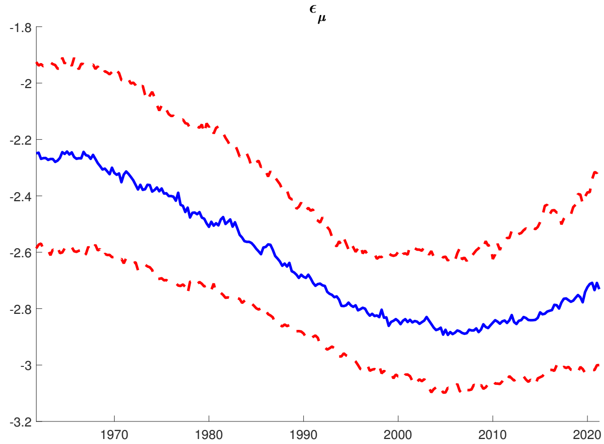
(d) Seasonal Factors γ_t

Figure 6: Canada: Estimated GDP components

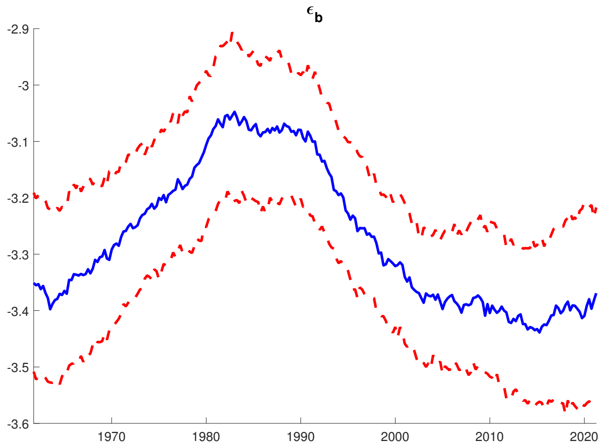
Given the previous results, Figure 7 depicts the significant changes in variances. In particular, variances from cyclical component and seasonal factors experimented a huge increase in the last portion of the sample, and this is not the case for the trend and intercept error variances. Trend and measurement error variances are in line with the Great Moderation hypoyhesis. Finally, the measurement error variance also exhibits a relevant change over time, which might be valid specially for the pandemic Covid-19 episode.



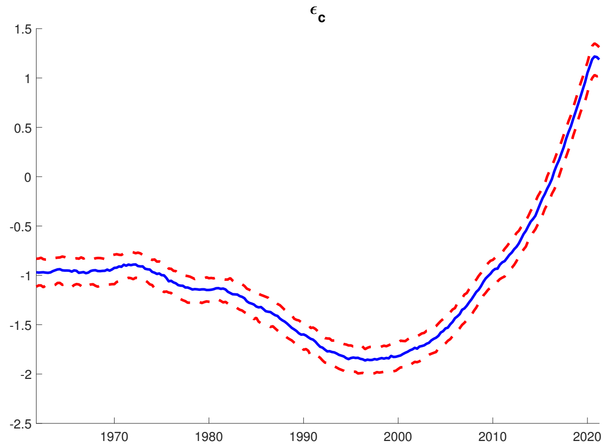
(a) Measurement Error



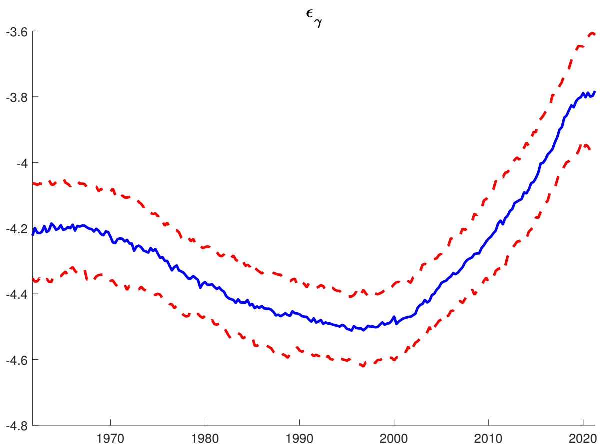
(b) Trend Error



(c) Intercept Error



(d) Cyclical Error c_t



(e) Seasonal Error γ_t

Figure 7: Canada: Estimated Log-Volatilities

4.3 Results for Peru

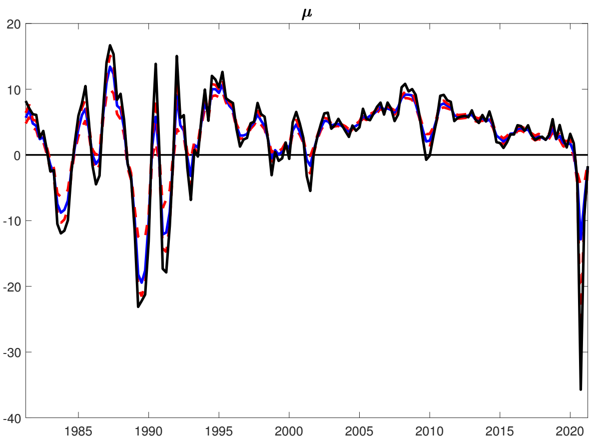
We finally turn to examine the results in Figure 8 using Peruvian GDP data, which is no seasonally adjusted. Estimated smooth trend component μ_t for Peru captures the structural change in 1980s associated with a big recession and a hyperinflation episode, as well as the recovery of the 1990s and the periods ahead. The Covid-19 pandemic episode deserves particular attention, where although the change is smooth, there is an evident structural break in the trend path.

Figure 8: Peru: Log-GDP and Trend component μ_t (1979.Q1-2021.Q1)

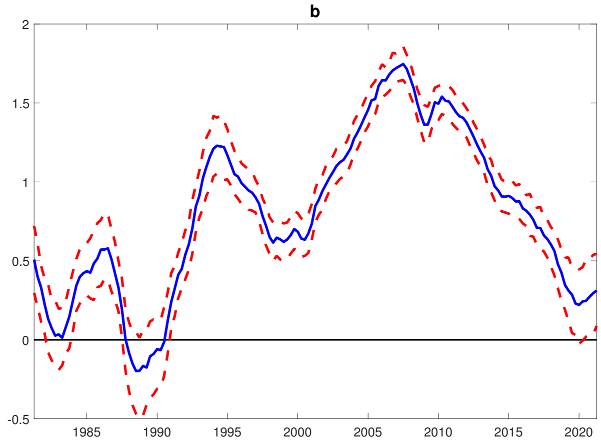
In line with the previous results, Figure 9 (panel (a)) compares the Year-to-Year growth rate of actual GDP and trend component μ_t. As it is expected, trend growth rate is smoother than the actual GDP. More interestingly, in this case we observe large movements both in trend μ_t [9] and in the cyclical component c_t (panel (c)) for the episodes considered as structural breaks such

[9] See Aguiar and Gopinath (2007) for discussion about the volatility in both trend and cycle components for emerging markets.

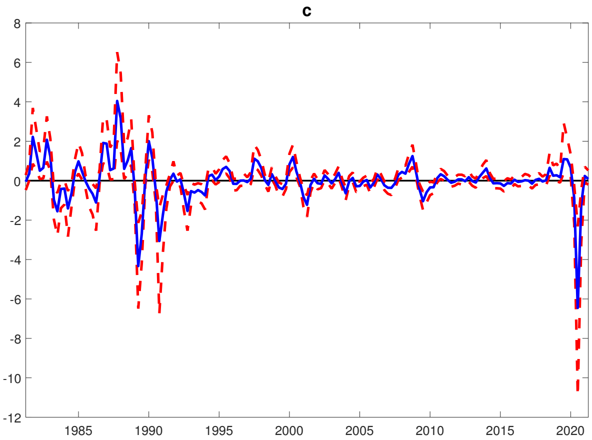
as the large recession with hyperinflation and also the Covid-19 pandemic. Finally, seasonal factors also seem to be well identified, with increasing magnitudes for the last portion of the sample.



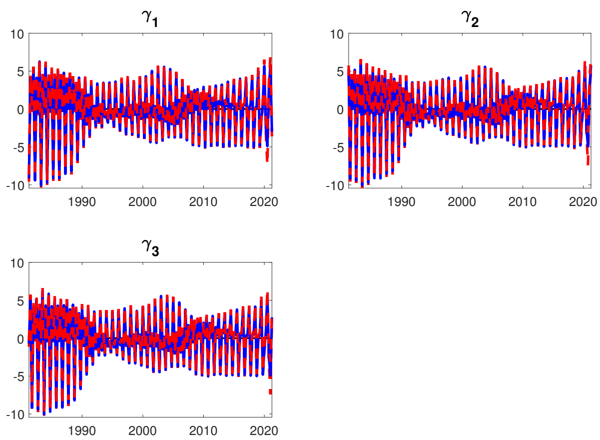
(a) Log-GDP and Trend μ_t YoY-Growth



(b) Time Varying Intercept b_t



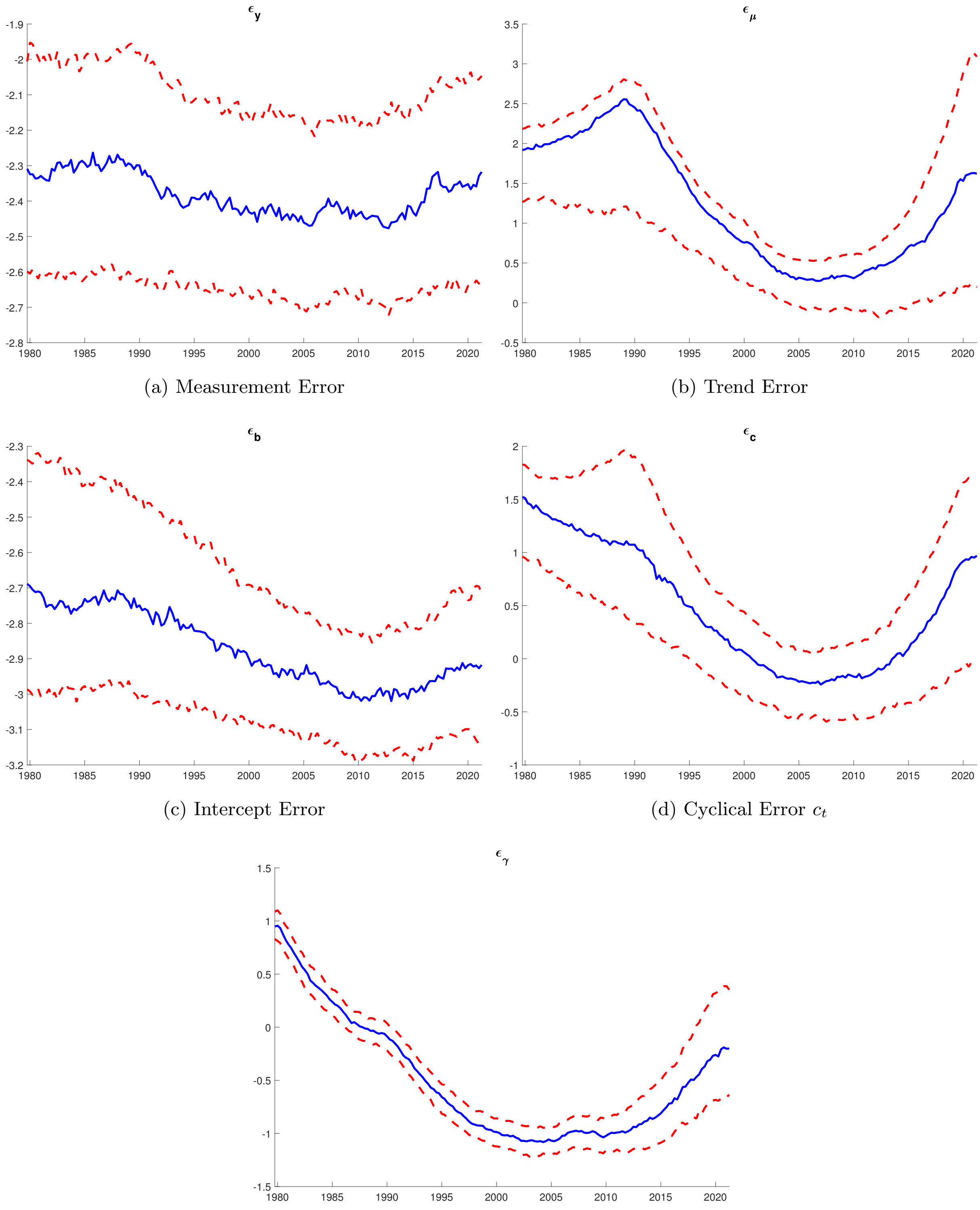
(c) Cyclical Component c_t



(d) Seasonal Factors γ_t

Figure 9: Peru: Estimated GDP components

Figure 10 suggest that there exist structural changes, especially for the cases of variances associated with trend and cyclical components, and for seasonal factors. That is, besides the moderation starting in the recovery period of 1990s, we observe an increase in variances in the last portion of the sample, which is mainly related with the Covid-19 pandemic. All in all, the identification of time series components is possible because of the consideration of stochastic volatility. We provide more evidence for this hypothesis in the next section.



(a) Measurement Error

(b) Trend Error

(c) Intercept Error

(d) Cyclical Error c_t

(e) Seasonal Error γ_t

Figure 10: Peru: Estimated Log-Volatilities

5 Model Comparison

Our Baseline specification considers the Stochastic Volatility component and two lags for the cyclical component, i.e. $p = 2$. We also consider alternative specifications without the time varying volatility feature and with only one lag, i.e. $p = 1$, with the purpose of comparing and determine which model describes better the data. To do so, a good practice in Bayesian Econometrics is to compute the Marginal Likelihood for each model (Chib, 2001). That is, we need to integrate out the posterior distribution across the parameter space, and the see to what extent a given model is a good representation of the data, i.e. the model with a higher marginal likelihood will be the best one. The marginal likelihood for each model M_i is

$$f\left(y^T \mid M_i\right) = \int L\left(\Theta_j \mid y^T, M_i\right) P\left(\Theta_j \mid M_i\right) dj \tag{19}$$

where the log-likelihood function is represented by equation (12). Given the scales, it is better to compute the log-marginal likelihood $\ln f\left(Y^T \mid M_i\right)$, and this is estimated using a standard harmonic mean estimator.

Model	Description	United States	Canada	Peru
M_1	With Stochastic Volatility and $p = 2$	-740.2	-686.5	-546.2
M_2	No Stochastic Volatility and $p = 2$	-1102.4	-947.8	-2292.0
M_3	With Stochastic Volatility and $p = 1$	-756.3	-691.4	-655.1
M_4	No Stochastic Volatility and $p = 1$	-1116.1	-1081.1	-2304.6

Table 1: Log-Marginal Likelihood $\ln f\left(Y^T \mid M_i\right)$

Given the results in Table 1, it is fairly easy to conclude that the model that has the best fit in the data is our benchmark approach (with the highest log-marginal density), and that any variation in the specification considered here delivers a poorer fit relative to the benchmark case. The consideration of stochastic volatility delivers extremely significant gains relative the constant variances case according to the Bayes's factor $BF_{i,j}$:

$$BF_{i,j} = exp\left(\ln f\left(Y^T \mid M_i\right) - \ln f\left(Y^T \mid M_j\right)\right) \tag{20}$$

6 Concluding Remarks

In this paper a so-called 'flexible filter' is proposed, which is of the Trend-Cycle type, but which considers shocks with time-varying variances (Stochastic Volatility). In general lines, the consideration of this component is a safe and flexible strategy against structural changes.

The results for United States, Canada and Peru adequately quantify the trend component of GDP, taking into account that this and other latent variables are subject to shocks whose variances change over time. The usefulness of the presented filter also lies in the fact that it quantifies the uncertainty associated with estimated latent factors and variances, which allows us to present confidence intervals easily.

Our research agenda is related with the application of a similar procedure to multivariate filters, i.e. semi-structural models and also richer models such as Dynamic Stochastic General Equilibrium (DSGE) models that can be taken to the data (see e.g. Diebold *et al.* (2017)).

A The State Space form of the system

Recall equations (8)-(9), the model re-written as a state-space system with time varying matrices:

$$y_t = D_t X_t + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H_t)$$

$$X_t = A_t X_{t-1} + R_t \eta_t, \qquad \eta_t \sim N(0, Q_t)$$

with $\Theta = \left(\theta, X^T\right)$ as the parameter set of the model and where $X_t = [\mu_t, c_t, c_{t-1}, b_t, \gamma_t']'$ is a $(2 + p + S - 1) \times 1$ vector. The matrix H_t is given by equation (7) with $k = y$. Then, given equation (1), with $p = 2$ and $S = 4$, we define $D_t = D_{t-1} = D$ as the matrix:

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{A.1}$$

Regarding the transition equation we define $A_t = A_{t-1} = A$ as the matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \phi_1 & \phi_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{A.2}$$

Then, Q_t is a (4×4) matrix, where its main diagonal is given by equation (7) with $k = \mu, c, b, \gamma$. Finally, $R_t = R_{t-1} = R$ is a rectangular (7×4) matrix given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.3}$$

B Gibbs Sampling details

1. **Block 1**: Sampling from $p\left(\varepsilon^{kT} \mid \Theta/\varepsilon^{kT}, Y^T\right)$: $k = (y, \mu, b, c, \gamma)$

   Given the parameter values of the model μ_t, c_t, b_t and γ_t and equations (1) – (2) – (4) – (5) – (6), compute the innovations η_t^k such that

$$\eta_t^y = y_t - \mu_t - c_t - \gamma_t \tag{B.1}$$

$$\eta_t^\mu = \mu_t - \mu_{t-1} - b_{t-1} \tag{B.2}$$

$$\eta_t^c = c_t - \phi_1 c_{t-1} - \phi_2 c_{t-2} \tag{B.3}$$

$$\eta_t^b = b_t - b_{t-1} \tag{B.4}$$

$$\eta_t^\gamma = \gamma_t + \sum_{s=1}^{S-1} \gamma_{t-1,s} \tag{B.5}$$

   Recall that $\eta_t^k \sim i.i.d.N\left(0, \varepsilon_t^k\right)$. In order to sample volatility ε_t^k we proceed for each k as follows :

$$ln\left(\left(\eta_t^k\right)^2 + 0.001\right) = ln\left(\varepsilon_t^k\right) + u_{k,t} \tag{B.6}$$

where $u_{k,t} \sim log\left(\chi^2\right)$ is approximated through a mixture of 7 normal distributions:

$$f\left(u_{k,t}\right) \approx \sum_{j=1}^{7} q_j f_N\left(u_{k,t} \mid m_j - 1.2704, \upsilon_j^2\right) \tag{B.7}$$

In addition, we have the transition equation

$$ln\left(\varepsilon_t^k\right) = ln\left(\varepsilon_{t-1}^k\right) + \epsilon_t^k, \qquad \epsilon_t^k \sim i.i.d.N\left(0, \sigma_k\right) \tag{B.8}$$

As a result, equations $(B.6) - (B.8)$ form a state-space system, so that we can simulate $ln\left(\varepsilon^k\right)^T$ following Kim *et al.* (1998), i.e. using the algorithm of Carter and Kohn (1994) conditional on the discrete variable s_k^T and given the prior (14).

2. **Block 2**: Sampling from $p\left(\phi \mid \Theta/\phi, y^T\right)$:

   Given the equation (4), we have that

$$c_t = \sum_{l=1}^{p} \phi_l c_{t-l} + \eta_t^c$$

   This expression can be re-arranged as a linear regression model:

$$c_t = \phi' \mathbf{Z}_t + \eta_t^c \tag{B.9}$$

   so that draws from the posterior distribution can be obtained from

$$\phi \mid c^T, \mathbf{\Sigma}^T \sim N\left(\overline{\phi}, \overline{V}\right) \times I\left(\phi\right) \tag{B.10}$$

   where $I\left(.\right)$ is the prior truncation for stationary draws.

and given the prior (16) we define

$$\overline{V} = \left[\underline{V}^{-1} + \sum_{t=p+1}^{T} \mathbf{Z}_t' \mathbf{\Sigma_t}^{-1} \mathbf{Z}_t \right]^{-1} \tag{B.11}$$

and

$$\overline{\phi} = \overline{V} \left[\underline{V}^{-1} \underline{\phi} + \sum_{t=p+1}^{T} \mathbf{Z}_t' \mathbf{\Sigma_t}^{-1} c_t \right], \tag{B.12}$$

with $\mathbf{\Sigma_t} = var\left(\eta_t^c\right)$ for ech $t = p+1, \ldots, T$.

3. **Block 3**: Sampling from $p\left(\sigma_k \mid \Theta/\sigma_k, y^T\right)$: $k = (y, \mu, b, c, \gamma)$

   Variance parameters are simulated using an Inverse-Gamma distribution. Given the prior $\sigma_k \sim IG\left(d_{\sigma_k^2} \times \underline{\sigma}_k, d_{\sigma_k}\right)$, the posterior distribution is:

$$p\left(\sigma_k \mid \Theta/\sigma_k, y^T\right) = IG\left(d_{\sigma_k} \times \underline{\sigma}_k + \sum_{t=2}^{T} u_{k,t}^2, d_{\sigma_k} + T - 1\right) \tag{B.13}$$

4. **Block 4**: Sampling from $p\left(X^T \mid \Theta/X^T, y^T\right)$: State-Space

   Given the State-Space form (8)-(9) with $X_t = [\mu_t, c_t, c_{t-1}, b_t, \gamma_t']'$ (see details in appendix A):

$$y_t = D_t X_t + \varepsilon_t, \qquad \varepsilon_t \sim N\left(0, H_t\right)$$

$$X_t = A_t X_{t-1} + R_t \eta_t, \qquad \eta_t \sim N\left(0, Q_t\right)$$

   Given the prior in (17) and the fact that the system is linear and normal, we sample the posterior distribution of X^T using the Kalman Filter and following Carter and Kohn (1994).

5. **Block 5**: Sampling from $p\left(s_k^T \mid \Theta/s_k^T, y^T\right)$: $k = (y, \mu, b, c, \gamma)$

   Conditional on the values of the other parameters, the term $\tilde{\varepsilon}_{k,t}^2$ is observable, so that we can sample the states, i.e. the elements of s_k^T independently from the following discrete distributions:

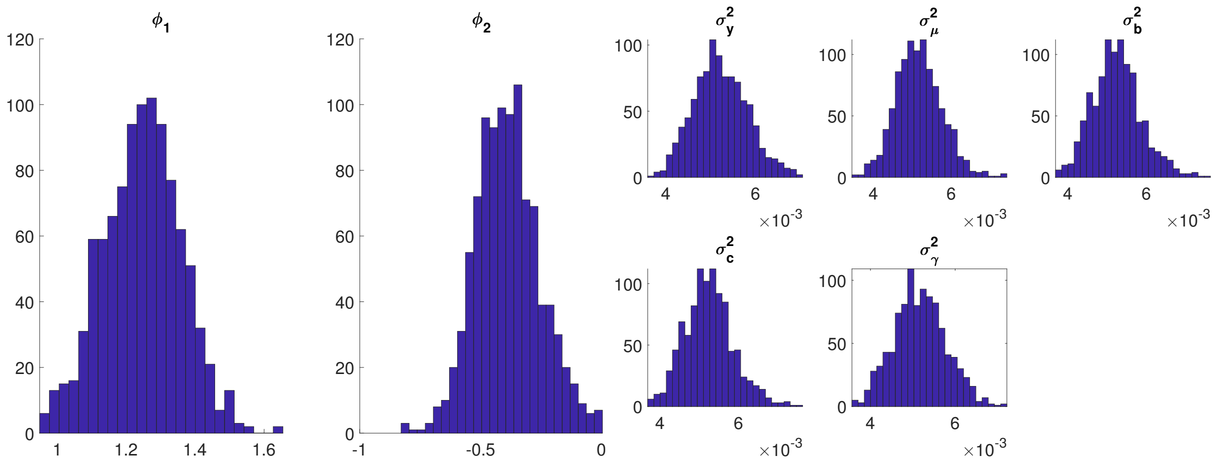
$$p\left(s_{k,t}=j \mid \tilde{\varepsilon}_{k,t}^{2}, \sigma_{k,t}^{2}\right) \propto f_{N}\left(ln\left(\tilde{\varepsilon}_{k,t}^{2}+0.001\right) \mid ln\sigma_{k,t}^{2}+m_{j}-1.2704, \upsilon_{j}^{2}\right) \tag{B.14}$$

where $j \in \{1,\ldots,7\}$ is the index for the state of the mixture of normals, $f_N\left(x \mid \mu, \sigma^2\right)$ is referred to the normal density function with mean μ, variance σ^2 and evaluated at point x. The values for means, variances and weights for the mixture of normals can be found in the following table.

s	$P(s=j)$	m_j	υ_j^2
1	0.00730	−10.12999	5.79596
2	0.10556	−3.97281	2.61369
3	0.00002	−8.56686	5.17950
4	0.04395	2.77786	0.16735
5	0.34001	0.61942	0.64009
6	0.24566	1.79518	0.34023
7	0.25750	−1.08819	1.26261

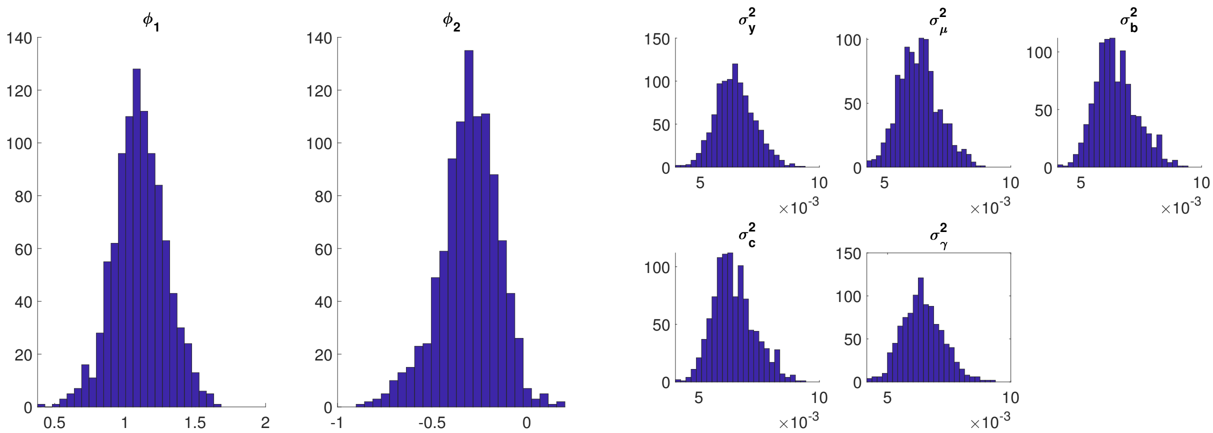
Table 2: $log\chi^2$ Distribution Approximation (Kim *et al.*, 1998)

C Additional Figures



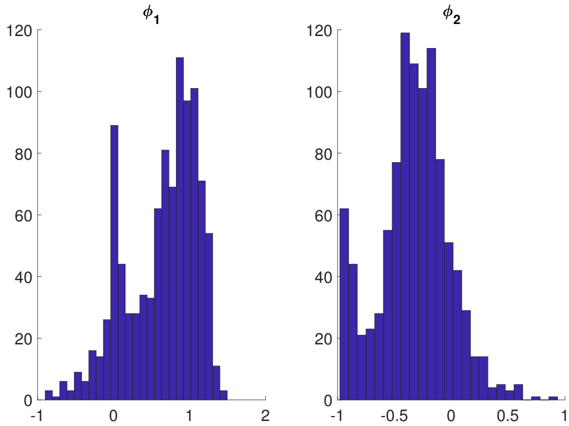
(a) Autorregresive Parameters (b) Variance Parameters

Figure C.11: United States: Posterior densities of Hyperparameters

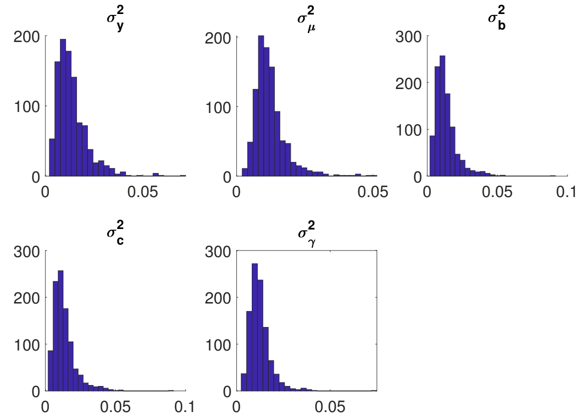


(a) Autorregresive Parameters (b) Variance Parameters

Figure C.12: Canada: Posterior densities of Hyperparameters



(a) Autorregresive Parameters



(b) Variance Parameters

Figure C.13: Peru: Posterior densities of Hyperparameters

References

AGUIAR, M. and GOPINATH, G. (2007). Emerging market business cycles: The cycle is the trend. *Journal of Political Economy*, **115** (1), 69–102, matlab code for program and data files attached.

ANDERSON, B. and MOORE, J. (1979). *Optimal Filtering*. Prentice-Hall, Inc.

BAXTER, M. and KING, R. (1999). Measuring business cycles: Approximate band-pass filters for economic time series. *The Review of Economics and Statistics*, **81** (4), 575–593.

BERNANKE, B. (2004). The great moderation, remarks at the meetings of the Eastern Economic Association, Washington, DC, February 20, 2004.

BEVERIDGE, S. and NELSON, C. R. (1981). A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the 'business cycle'. *Journal of Monetary Economics*, **7**, 151–174.

CANOVA, F. (1998). Detrending and business cycle facts. *Journal of Monetary Economics*, **41**, 475–512.

— (2007). *Methods for Applied Macroeconomic Research*. Princeton University Press.

— (2020). *FAQ: How do I measure the Output gap?* CEPR Discussion Papers 14943, C.E.P.R. Discussion Papers.

CARRIERO, A., CLARK, T. E. and MARCELLINO, M. (2016). Common drifting volatility in large bayesian vars. *Journal of Business and Economic Statistics*, **34** (3), 375–390.

CARTER, C. K. and KOHN, R. (1994). On gibbs sampling for state space models. *Biometrika*, **81** (3), 541–553.

CHIB, S. (2001). Markov chain monte carlo methods: computation and inference. In J. Heckman and E. Leamer (eds.), *Handbook of Econometrics*, vol. 5, *57*, 1st edn., Elsevier, pp. 3569–3649.

CHRISTIANO, L. J. and FITZGERALD, T. J. (2003). The band pass filter. *International Economic Review*, **44** (2), 435–465.

Clark, P. (1987). The cyclical component of u. s. economic activity. *The Quarterly Journal of Economics*, **102** (2), 797–814.

Cogley, T. and Nason, J. M. (1995). Effects of the hodrick-prescott filter on trend and difference stationary time series implications for business cycle research. *Journal of Economic Dynamics and Control*, **19** (1), 253–278.

Del Negro, M. and Primiceri, G. (2015). Time varying structural vector autoregressions and monetary policy: A corrigendum. *Review of Economic Studies*, **82**, 1342–1345.

Diebold, F., Schorfheide, F. and Shin, M. (2017). Real-time forecast evaluation of dsge models with stochastic volatility. *Journal of Econometrics*, **201**, 322–332.

Durbin, J. and Koopman, S. (2002). A simple and efficient simulation smoother for state space time series analysis. *Biometrika*, **89** (3), 603–615.

Enders, W. and Li, J. (2015). Trend-cycle decomposition allowing for multiple smooth structural changes in the trend of us real gdp. *Journal of Macroeconomics*, **44**, 71–81.

Hamilton, J. (1989). A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica*, **57** (2), 357–384.

— (2018). Why you should never use the hodrick-prescott filter. *The Review of Economics and Statistics*, **100** (2), 831–843.

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton University Press.

Harvey, A. (1985). Trends and cycles in macroeconomic time series. *Journal of Business and Economic Statistics*, **3** (3), 216–227.

— and Trimbur, T. (2003). General model-based filters for extracting cycles and trends in economic time series. *The Review of Economics and Statistics*, **85**, 244–255.

— and — (2007). Trend estimation and the hodrick-prescott filter. *Journal of the Japan Statistical Society*, **38** (1), 41–49.

Harvey, A. C. (1989). *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge University Press.

Herbst, E. P. and Schorfheide, F. (2016). *Bayesian Estimation of DSGE Models*. Princeton University Press.

Hodrick, R. and Prescott, E. (1997). Postwar u.s. business cycles: An empirical investigation. *Journal of Money, Credit and Banking*, **29** (1), 1–16.

Hodrick, R. J. (2020). An exploration of trend-cycle decomposition methodologies in simulated data, nBER Working papers No. 26750.

Jacquier, E., Polson, N. G. and Rossi, P. E. (1994). Bayesian analysis of stochastic volatility models. *Journal of Business and Economic Statistics*, **20** (1), 69–87.

Kim, C.-J. and Nelson, C. (1999). *State-Space Models with Regime Switching: Classical and Gibbs-Sampling Approaches with Applications*. The MIT Press.

Kim, S., Shephard, N. and Chib, S. (1998). Stochastic volatility: Likelihood inference and comparison with ARCH models. *The Review of Economic Studies*, **65** (3), 361–393.

Luo, S. and Startz, R. (2014). Is it one break or ongoing permanent shocks that explains u.s. real gdp? *Journal of Monetary Economics*, **66**, 155–163.

Lütkepohl, H. (2005). *New Introduction to Multiple Time Series Analysis*. Springer.

Maravall, A. and Del Río, A. (2001). Time aggregation and the hodrick-prescott filter, working Papers 0108, Banco de España.

Morley, J., Nelson, C. and Zivot, E. (2003). Why are the beveridge-nelson and unobserved-components decompositions of gdp so different? *The Review of Economics and Statistics*, **85** (2), 235–243.

Nelson, C. R. and Plosser, C. I. (1982). Trends and random walks in macroeconomic time series: Some evidence and implications. *Journal of Monetary Economics*, **10**, 139–162.

PELAGATTI, M. (2016). *Time series modelling with unobserved components.* CRC Press, Taylor and Francis.

PERRON, P. and WADA, T. (2016). Measuring business cycles with structural breaks and outliers: Applications to international data. *Research in Economics*, **70**, 281–303.

— and YABU, T. (2009). Testing for shifts in trend with an integrated or stationary noise component. *Journal of Business and Economic Statistics*, **27** (3), 369–396.

STOCK, J. and WATSON, M. (2003). Has the business cycle changed?, paper presented at the Federal Reserve Bank of Kansas City "Monetary Policy and Uncertainty: Adapting to a Changing Economy" symposium Jackson Hole, WY, August 28-30, 2003.