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Fabrizio Orrego\*

\* Carnegie Mellon University and Central Bank of Peru.

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# Habit formation and sunspots in overlapping generations models\*

Fabrizio Orrego<sup>†</sup>  
Carnegie Mellon University  
Central Bank of Peru

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## Abstract

I introduce habit formation into an otherwise standard overlapping generations economy with pure exchange populated by three-period-lived agents. Habits are modeled in such a way that current consumption increases the marginal utility of future consumption. With logarithmic utility functions, I demonstrate that habit formation may give rise to stable monetary steady states in economies with hump-shaped endowment profiles and reasonably high discount factors. Intuitively, habits imply adjacent complementarity in consumption, which in turn helps explain why income effects are sufficiently strong in spite of logarithmic utility. The longer horizon further strengthens the income effect. Finally, I use the bootstrap method to construct stationary sunspot equilibria for those economies in which the steady state is locally stable.

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<sup>†</sup>Email address: [forrego@andrew.cmu.edu](mailto:forrego@andrew.cmu.edu)

# 1 Introduction

In textbook models of overlapping generations (OLG), sunspot equilibria have been intimately related to backward bending offer curves and stable monetary steady states (see Blanchard and Fischer [6]). Theoretically, if income effects are strong enough, then aggregate savings is a strictly decreasing function of the real return and the monetary steady state is a sink. In two-period models equipped with power utility, such income effects require arguably unrealistic parameter specifications (discount factor close to zero, income concentrated in the first period of life and high curvature of utility function), which may explain the apparent reluctance of the profession to embrace the idea of a sunspot equilibrium.

In this paper I introduce habit formation into an otherwise standard three-period-lived OLG economy, in an attempt to overcome the legitimate concern discussed above. I show that the steady state is stable when the discount factor is close to one-half, the endowment profile is hump-shaped and the utility function is logarithmic. Then I use the forward stability of the steady-state dynamics to construct local sunspot equilibria around the steady state, as in Spear, Srivastava and Woodford [24].

In light of recent economic developments, the fresh perspective on stability presented in this paper can also provide an alternative explanation, via the sunspot equilibria, for the excess volatility that is currently observed, or for the fact that the *great moderation* in macroeconomic variability in the U.S. may well be part of an on-going cyclic equilibrium variation<sup>1</sup>.

I assume that agents derive utility from the gap between current and past consumption. In presence of subtractive habits not only the elasticity of intertemporal substitution is less than one at any steady state, but also dated consumption goods are weak gross complements at yields near the golden rule. I show that the set of economies in which the steady state is a sink is relatively large when the discount factor is close to zero, but shrinks gradually as the discount factor approaches one-half. Intuitively, when the discount factor is small enough, the desire for youth consumption increases, which affects positively the marginal utility of consumption in the second stage and fosters adult consumption. This in turn increases the marginal utility of consumption in the third stage and so on. The adjacent complementarity in consumption helps explain why income effects are sufficiently strong in spite of the logarithmic utility.

On the other hand, the longer horizon further strengthens the income effect and helps support stable steady states for reasonable values of the discount factor. The intuition is as follows. When the monetary steady state is stable, the equilibrium sequence of returns converges to the golden rule via damped oscillations. Note that in the three-period case, households demand assets when young and middle-aged. If returns are high today, then the current middle-age agents' wealth is small, since they received low returns on their savings

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<sup>1</sup>I thank Stephen Spear for this idea.

yesterday. Consequently, their demand for money is low. This in turn suggests that, even if the young agents' demand for money is high because of the high returns, the aggregate demand for savings can be low because the middle-age agents are constrained by their wealth. Thus, because of the wealth distribution among different generations, the three-period model possesses a mechanism for causing aggregate asset demand to decrease with the current rate of return that is completely absent from the two-period model.

The remainder of the paper is structured as follows. Section 2 introduces the layout of the model. Section 3 derives the competitive equilibrium. Section 4 deals with the competitive equilibrium with sunspots. The last section concludes.

## Related literature

Under the condition that the monetary steady state is stable, there can be a continuum of convergent solutions, as well as cyclic solutions. In the former situation, I say that the competitive equilibrium is indeterminate<sup>2</sup>. Kehoe and Levine [18] show robust examples of indeterminacy in three-period-lived OLG economies with outside money if the discount factor is one-half, income is hump-shaped and the intertemporal elasticity of substitution is one fourth<sup>3</sup>.

An important property of an intertemporal elasticity of substitution of one-fourth is that it allows goods in different periods to be gross complements at some prices. In situations where all goods are gross substitutes, Kehoe, Levine, Mas-Colell and Woodford [19] show that indeterminacy of the type discussed above is impossible.

Theoretical foundations for several versions of intrinsic habit formation have been recently provided by Rozen [22]. In the OLG framework, habit formation creates room for indeterminacy and is open to a finite number of different specifications, with diverse implications on asset prices. To the best of my knowledge, Lahiri and Puhakka [20] first incorporate subtractive habit formation in a two-period OLG model with outside money. They show that cycles arise when the discount factor is near zero and utility is logarithmic, because generational offer curves must bend backwards (however, later on in this paper I establish that the offer curve does not bend backwards in their model).

The current three-period extension of Lahiri and Puhakka [20] is by no means trivial, as several authors have demonstrated that an economy in which the life cycle goes to three

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<sup>2</sup>On the other hand, Bhattacharya and Russell [5] show that in three-period OLG economies there exist two-period cycles if the elasticity of substitution is less than one-fourth and income is positive only in the first period of life, for any value of the discount factor.

<sup>3</sup>Unfortunately, it remains difficult to support indeterminate equilibria in multiperiod OLG economies with pure exchange, outside money and more complex structures. For instance, Azariadis and Lambertini [2] find that indeterminacy arises in three-period economies in the presence of endogenous borrowing constraints, logarithmic utility functions and hump-shaped endowment profiles. However, their result is not robust to the introduction of unproductive assets in positive net supply.

periods or more may behave differently from the well-known Diamond model (see for example Azariadis, Lee and Ohanian [3], Kehoe and Levine [18] and Henriksen and Spear [16]).

I use in this paper the bootstrap method studied by Spear, Srivastava and Woodford [24] to construct stationary rational expectations equilibria with sunspots, which are known to exist in OLG models according to the Philadelphia Pholk Theorem<sup>4</sup>. In the derivation of these equilibria, I find that prices depend on lagged endogenous variables, as in Henriksen and Spear [16]. This result was first presented by Spear, Srivastava and Woodford [24] in a two-period model with  $n > 1$  goods and  $m > 1$  agents. The similarity is not surprising, because the three-period-lived model may also be interpreted as a two-good, two-consumer, two-period-lived model (see Balasko, Cass and Shell [4] and Kehoe and Levine [18]). In this paper I stick to the former specification since it drastically reduces the number of parameters that are not subject to normalization and therefore the analysis remains as simple as possible.

## 2 The model

I study a closed, pure-exchange economy of overlapping generations of agents who live for three periods, labeled as young adulthood, middle age and retirement. There is one individual in each cohort and no population growth. Time is discrete. In period 1, in addition to the young consumer, there are two others, an old consumer who lives in period 1 only and a middle-aged consumer who lives in periods 1 and 2. Agents are allocated exogenous amounts of the unique consumption good in each period of life. I assume, as in Geanakoplos, Magill and Quinzii [14], that these endowments are stationary, and the typical young agent's lifetime endowment is  $\omega = (\omega_1, \omega_2, 0) \in \mathbb{R}_+^3$ , which can be interpreted as the agent's labor income (income in retirement is zero).

Agents are allowed to trade a costlessly storable asset called money to redistribute income over time. In period 1, the initial old and middle-aged receive fixed money endowments  $\bar{m}_{2,1}$  and  $\bar{m}_{3,1}$ , respectively, such that  $\bar{m}_{2,1} + \bar{m}_{3,1} = \bar{m}$  (young agents receive no endowments of money). Here, as in the rest of the paper,  $x_{i,j}$  denotes the value of  $x$  at time  $j$  when the agent is in the  $i$ -th stage of life, for  $i = 1, 2, 3$ . The initial young will demand  $m_{1,t}$  amounts of money today and  $m_{2,t+1}$  tomorrow and, as time passes by, both adults and old agents have money endowments in the amounts carried forward from the previous period. In this economy, the role of government (or central bank) is to keep the aggregate supply of money fixed at  $\bar{m}$  at all times.

Lifetime preferences for consumption of a typical young agent are specified by the following utility function:

$$U [c_{1,t}, c_{2,t+1}, c_{3,t+2}] = \ln (c_{1,t}) + \beta \ln (c_{2,t+1} - \gamma c_{1,t}) + \beta^2 \ln (c_{3,t+2} - \gamma c_{2,t+1}) \quad (1)$$

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<sup>4</sup>It is also possible to construct sunspot equilibria of the Markovian variety first examined by Azariadis [1].

where  $c = (c_{1,t}, c_{2,t+1}, c_{3,t+2})$  is the consumption over the life cycle,  $\beta \in (0, 1]$  is the discount factor and  $\gamma \in \mathbb{R}_+$  is the habit parameter. As in Lahiri and Puhakka [20], agents derive utility from the gap between current and past consumption. Note that habits vanish completely after one period of twenty years.

I work with subtractive habits in (1), because this representation has several advantages over the multiplicative version also used in the literature (e.g. Bunzel [8])<sup>5</sup>. For instance, the properties of the Hessian of  $U[c]$  remain qualitatively intact after the introduction of habits (in particular, the negative semidefiniteness of the Hessian of  $U[c]$ ). However, it may be necessary to impose certain restrictions on the Jacobian of  $U[c]$  to prevent the intertemporal marginal rates of substitution from being negative, as subtractive habit formation may cause consumption goods to become consumption bads. Specifically, the restrictions needed are as follows:

$$\begin{aligned} -\frac{dc_{2,t+1}}{dc_{1,t}} &= \frac{c_{1,t}^{-1} - \beta\gamma(c_{2,t+1} - \gamma c_{1,t})^{-1}}{\beta(c_{2,t+1} - \gamma c_{1,t})^{-1} - \beta^2\gamma(c_{3,t+2} - \gamma c_{2,t+1})^{-1}} > 0 \\ -\frac{dc_{3,t+2}}{dc_{2,t+1}} &= \frac{(c_{2,t+1} - \gamma c_{1,t})^{-1} - \beta\gamma(c_{3,t+2} - \gamma c_{2,t+1})^{-1}}{\beta(c_{3,t+2} - \gamma c_{2,t+1})^{-1}} > 0 \end{aligned} \tag{2}$$

It turns out that these two conditions can be readily disregarded in this paper, since the homotheticity of preferences (implied by the logarithmic specification) is preserved under subtractive habits (see Bossi and Gomis-Porqueras [7] for an algebraic proof), and hence all goods remain normal.

Let  $p_t$  be the price level of the consumption good (I use the price of money as the numeraire). If conditions (2) hold, then the agent chooses  $c$  and money holdings  $m = (m_{1,t}, m_{2,t+1})$  so as to maximize (1), subject to the sequential budget constraints:

$$\begin{aligned} p_t c_{1,t} &\leq p_t \omega_1 - m_{1,t} \\ p_{t+1} c_{2,t+1} &\leq p_{t+1} \omega_2 + m_{1,t} - m_{2,t+1} \\ p_{t+2} c_{3,t+2} &\leq m_{2,t+1} \end{aligned} \tag{3}$$

plus the usual non-negativity constraints on consumption and *consumption services* (defined as  $\tilde{c}_{2,t+1} = c_{2,t+1} - \gamma c_{1,t}$  and  $\tilde{c}_{3,t+2} = c_{3,t+2} - \gamma c_{2,t+1}$ ). Note that in this model  $m_{2,t+1} > 0$ , because middle-age agents must save to finance consumption when old, but  $m_{1,t}$  could be either positive (i.e. asset) or negative (i.e. liability). Finally, because  $U[c]$  is continuous and the constrained choice set is compact, the consumer's problem has a solution by a routine

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<sup>5</sup>In the subtractive version, the instantaneous utility function depends on the distance between current consumption and past consumption; in the multiplicative approach, the utility is a function of the ratio of current consumption to past consumption.

application of the Weierstrass theorem.

### 3 Perfect-foresight equilibrium

In order to build the main argument of the paper, I first assume that agents have perfect foresight, which is the rational expectations equilibrium requirement in a deterministic setting. To begin with, I define a competitive equilibrium for this economy with outside money. Notice that in this definition the initial generations have been conveniently disregarded.

**Definition 1** *A monetary competitive equilibrium in this economy is a sequence of consumption bundles  $\{c_{1,t}, c_{2,t+1}, c_{3,t+2}\}_{t=1}^{\infty}$ , money claims  $\{m_{1,t}, m_{2,t+1}\}_{t=1}^{\infty}$  and prices  $\{p_t\}_{t=1}^{\infty}$ , such that: (a) given prices, the agent born at an arbitrary period  $t$  maximizes (1) subject to (2)-(3), plus the usual non-negativity constraints on consumption and consumption services, and (b) the money market clears  $\forall t$ , i.e.  $m_{1,t} + m_{2,t} = \bar{m}$ .*

The first part of the definition can be handled easily by a routine application of the Kuhn-Tucker theorem under convexity. However, a more direct approach is as follows. I guess that both  $\tilde{c}_{2,t+1} > 0$  and  $\tilde{c}_{3,t+2} > 0$  hold and solve the problem using the Lagrange Theorem. From the (necessary and sufficient) first-order conditions for an interior optimum, it can be shown that the unique solution is given by:

$$c_{1,t} [p_{t+2}, p_{t+1}, p_t] = \frac{p_t \omega_1 + p_{t+1} \omega_2}{(1 + \beta + \beta^2) (p_t + \gamma p_{t+1} + \gamma^2 p_{t+2})} \quad (4)$$

$$c_{2,t+1} [p_{t+2}, p_{t+1}, p_t] = \left\{ (1 + \beta) \gamma + \frac{\beta p_t}{p_{t+1} + \gamma p_{t+2}} \right\} c_{1,t} \quad (5)$$

Finally, I verify that (4)-(5) satisfy both (2)-(3), as well as the non-negativity constraints on consumption and *consumption services* and hence the initial guess is verified.

With (4)-(5) I write the market-clearing condition as a non-linear function:

$$F_0 [p_{t+2}, p_{t+1}, p_t, p_{t-1}] = 0$$

Because preferences are represented by a logarithmic utility function, it is possible to get a closed-form expression for  $p_{t+2}$ . This is the content of the following proposition:

**Proposition 1** *The perfect-foresight dynamics of prices are described by a third order non-linear difference equation of the form:*

$$p_{t+2} = F_1 [p_{t+1}, p_t, p_{t-1}] \quad (6)$$

**Proof.** See appendix. ■

With this equation at hand, it is convenient to reformulate the original definition of a competitive equilibrium as follows:

**Definition 2** *A perfect-foresight equilibrium can also be viewed as a forecast function  $F_1$  which satisfies the first-order conditions (for optimality) and the market-clearing condition.*

Now I use (6) to study the existence and uniqueness of a stationary equilibrium, in which prices (and consumption bundles) remain constant over time. Stationary variables do not have time subscripts. The results are summarized in the following proposition:

**Proposition 2** *There exists a unique steady-state price  $p$ , which satisfies  $m_1 [p] + m_2 [p] - \bar{m} = 0$  for endowment  $\omega$  which is a monetary equilibrium.*

**Proof.** See appendix. ■

As is standard in OLG models with perfect foresight, this proposition does not imply positive aggregate savings in equilibrium. Since the endowment profile attached to each newborn is  $\omega = (\omega_1, \omega_2, 0)$ , adults must save in equilibrium to finance consumption when old. However, young individuals save if  $\omega_1$  is relatively high, otherwise they prefer to transfer debt from one generation to another so as to smooth consumption. Therefore, I assume from now on the following constraint on  $\omega$  in order to guarantee positive aggregate savings in equilibrium:

**Assumption 1** *The endowment  $\omega$  is such that  $(2 - \lambda)\omega_1 + (1 - \lambda)\omega_2 > 0$ , where  $\lambda = \frac{(1+\beta)(1+\gamma+\gamma^2)+1+2\gamma}{(1+\beta+\beta^2)(1+\gamma+\gamma^2)(1+\gamma)}$ , and hence the monetary steady state with equilibrium price  $p > 0$  has aggregate savings equal to  $\bar{m} > 0$ .*

Note that the condition on  $\omega$  is easily fulfilled with a low value of  $\omega_1$  when either the discount factor or the habit parameter is high enough. Intuitively, if young agents become more patient, then other things equal they are more willing to save out of their endowment. Also, young agents save more as the habit effect strengthens, other things equal, as youth consumption puts a floor on adult consumption.

Now I convert the third-order difference equation defined by (6) into a first-order non-linear system of the form:

$$q_{t+2} = F_2 [q_{t+1}]$$

where  $q_{t+1} = (p_{t+1}, p_t, p_{t-1})'$  and  $F_2$  is given by the equation:

$$\begin{pmatrix} p_{t+2} \\ p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} F_1 [p_{t+1}, p_t, p_{t-1}] \\ p_{t+1} \\ p_t \end{pmatrix}$$



At  $p$ , the Jacobian matrix associated to this system is given by a 3x3 matrix  $G$  of the form:

$$G = \begin{pmatrix} \partial F_1 [p] / \partial p_{t+1} & \partial F_1 [p] / \partial p_t & \partial F_1 [p] / \partial p_{t-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The Hartman-Grobman theorem asserts that if  $p$  is a hyperbolic steady state of (6) and if the Jacobian matrix  $G$  is invertible (which is indeed the case since  $G$  is full rank), then there is a neighborhood  $\mathbf{P}$  of  $p$  in which the first-order Taylor series expansion with Jacobian matrix  $G$  is topologically equivalent to the original non-linear system.

Generically, there is no need to worry about non-hyperbolic equilibria. Given an arbitrary system  $x_t = F [x_{t-1}, a]$  with  $F [\cdot, a] \in C^1$ , the moduli of eigenvalues are continuous functions of the parameters  $a$ . If the steady state of  $F [\cdot, a]$  is non-hyperbolic at  $a = a_0$ , then generically any small change in the value of  $a$  yields a hyperbolic equilibrium close to the old one, provided that the steady state itself does not disappear (see de la Fuente [12]).

Let  $\#e$  denote the number of eigenvalues of  $G$  whose moduli lie outside the unit circle. As there are two non-predetermined variables in (6), namely  $p_{t+1}$  and  $p_t$ , the Blanchard-Kahn eigenvalue conditions, which are essential to characterize equilibrium dynamics, are summarized in the following proposition:

**Proposition 3 (Blanchard-Kahn)** *Let  $p$  be a hyperbolic steady state of (6) and let the Jacobian matrix  $G$  be invertible. If the number of eigenvalues of  $G$  whose moduli lie outside the unit circle is*

1. equal to two, then the steady state is locally saddle-path stable, i.e. there exists a unique competitive equilibrium (determinacy).
2. greater than two, then there is no non-explosive solution satisfying (6).
3. less than two, then the steady state is locally stable, i.e. there is an infinity of competitive equilibria (indeterminacy).

In the presence of habit formation, an economy is completely characterized by the following set of parameters  $(\omega_1, \omega_2, \gamma, \beta)$ . I assume that  $\gamma \in [0, 1]$ , because the habit parameter typically belongs to the unit interval in the calibration exercises performed by Bossi and Gomis-Porqueras [7]. I further assume that  $\omega_1 + \omega_2 = 1$ , without loss of generality, since homotheticity is preserved under subtractive habit persistence. With these simplifications, an economy can be represented by a triplet  $(\omega_1, \gamma, \beta) \in [0, 1] \times [0, 1] \times (0, 1]$ .

In Figure 1, I depict the  $(\omega_1, \gamma)$  Cartesian plane for four possible values of  $\beta$ , and assess the stability properties of every hyperbolic equilibrium, following the Blanchard-Kahn eigenvalue

conditions:

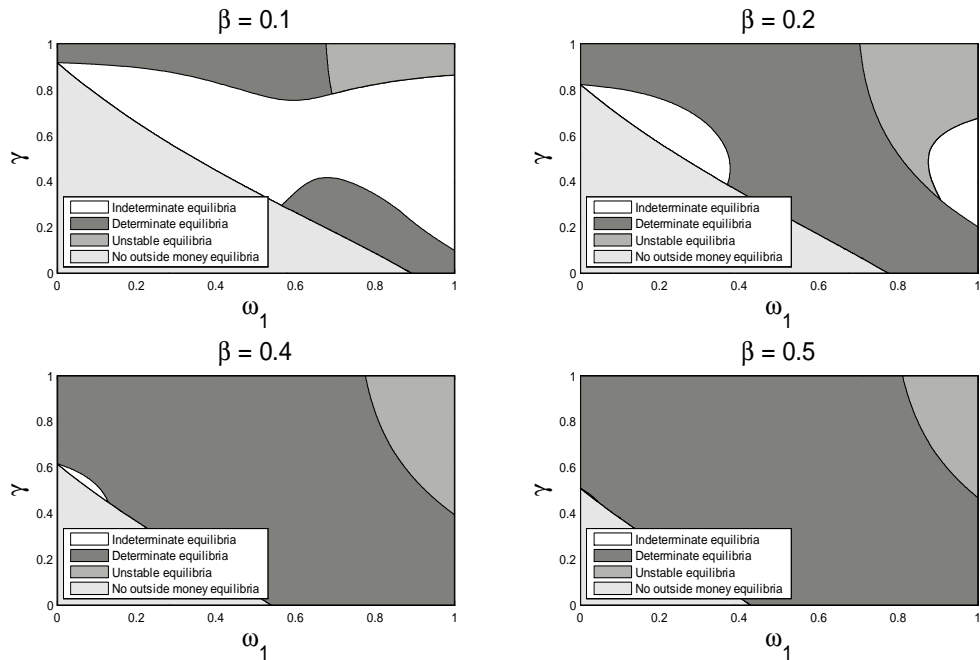


Figure 1. Stability properties of the monetary steady state for different parameter configurations  $(\omega_1, 1 - \omega_1, \gamma, \cdot)$  and  $\beta \in \{0.1, 0.2, 0.4, 0.5\}$ . At every steady state, *indeterminacy* occurs when there is more than one stable eigenvalue, *determinacy* occurs when there is one stable eigenvalue, *instability* occurs when there is less than one stable eigenvalue and, finally, *no outside money equilibrium* means that Assumption 1 is not satisfied.

First of all, the discount factor is smaller than in representative-agent models. These values are not implausible, however, because each period in OLG models of this sort typically consists of 20 years. For example, Kehoe and Levine [18] and Geanakoplos, Magill and Quinzii [14] set the discount factor to  $0.5 = (0.96594)^{20}$ , while Constantinides, Donaldson and Mehra [11] assume the discount factor is 0.44.

The routine builds  $G$  at every possible coordinate pair in the grid, conditional on Assumption 1 being satisfied, and then calculates  $\#e$ . Since  $F_1$  is provided explicitly in the appendix, the exact elements of  $G$  are computed in each round.

There are four different colors in each panel of figure one: light gray, gray, dark gray and white. It is well known that if the original equilibrium is hyperbolic, then small changes in  $\gamma$  do not take the moduli of eigenvalues across the boundary of the unit circle. However, in

Figure 1 there are small perturbations of  $\gamma$  near certain critical values that change the nature of the steady state. Then, bifurcations as defined by Guckenheimer and Holmes [15] occur at non-hyperbolic equilibria in this model.

To begin with, notice that all economies located in the light gray area do not admit a steady state with positive aggregate savings. In these economies, the assumption on the endowment profile  $\omega$  is not met. As expected, equilibria with outside money are not well defined when the endowment of the young is low enough.

In the gray area, the steady state is unstable as  $\#e = 3$ . However, this case may be disregarded because instability requires that  $\omega_1 > \omega_2$ , which is at odds with what is observed in reasonable calibration exercises. Conversely, the stationary economies located in the dark gray areas are locally saddle-path stable, because  $\#e = 2$ . Azariadis, Lee and Ohanian [3] show that if habits are absent in a model like mine, then this is the only possible outcome.

The introduction of habit persistence, nevertheless, makes the steady state locally stable in all economies located in the white areas, as  $\#e < 2$ . Indeterminate equilibria are consistent with hump-shaped endowment profiles and reasonably high discount factors. In particular, the economy characterized by  $\theta \equiv (\omega_1, \omega_2, \omega_3, \gamma, \beta) = (0.1, 0.9, 0, 0.5, 0.4)$  satisfies the eigenvalue criterion<sup>6</sup>. Thus I have proved the following proposition:

**Proposition 4** *For a non-empty, open set of economies satisfying Assumption 1, there exists a neighborhood  $\mathbf{P}$  of  $p$  such that the map  $F_2$  is a contraction on  $\mathbf{P}(p) \times \mathbf{P}(p) \times \mathbf{P}(p)$ .*

### Origins of indeterminacy: Interpretation

First, it is convenient to discuss the behavior of the offer curve. Let  $A_t = \alpha_{1,t} + \alpha_{2,t}$  denote aggregate savings, where  $(\alpha_{1,t}, \alpha_{2,t+1}) = (m_{1,t}/p_t, m_{2,t+1}/p_{t+1})$  represent the assets held by the young and middle-aged at the end of periods  $t$  and  $t + 1$ , respectively. Typically, the offer curve uses the market-clearing condition to relate  $A_{t+1}$  to  $A_t$  in the  $(A_t, A_{t+1})$  plane. Now the asset market clears when  $A_t = \bar{m}/p_t$  or  $A_{t+1} = R_t A_t$ , where  $R_t = p_t/p_{t+1}$  is the gross rate of return<sup>7</sup>.

Notice that the set of economies in which the steady state is a sink is relatively large when the discount factor is close to zero, but disappears gradually as the discount factor approaches one-half. Intuitively, when the discount factor is small enough, the desire for youth consumption increases, which affects positively the marginal utility of consumption in the second stage and fosters adult consumption. This in turn increases the marginal utility of consumption in the third stage and so on. Put it differently, indeterminacy arises when

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<sup>6</sup>This parameterization suggests that agents get 90 percent of their lifetime income when middle-aged. In the literature, it is possible to encounter similar specifications, for instance, if agents earn labor income only in the second period of their life, as in Jappelli and Pagano [17].

<sup>7</sup>As in Blanchard and Fischer [6], a monetary equilibrium could be defined, in principle, without direct reference to the stock of currency.

agents are impatient and dated consumption goods become weak gross complements at yields near the golden rule.

The longer horizon adds an additional effect on the aggregate saving function. Because of Proposition 4, there is an open set of economies in which the offer curve crosses the 45-degree line from below with a slope between  $-1$  and  $0$ , such that  $A_t = A_{t+1} = A > 0$ . This means that the equilibrium sequences of returns converge to the monetary steady state via damped oscillations, as the slope of any straight line that goes through the origin equals  $R_t$ . That is, returns switch between high and low and converge to 1.

Now suppose returns are high at  $t$ . Hence the current middle-age agents' wealth is small, as they received low returns on their savings at  $t - 1$ . Consequently, their demand for money is low at time  $t$ . Therefore, even if the young agents's demand for money is high because of the high returns, the aggregate demand for savings can be low because the middle-aged are constrained by their wealth. Thus, with longer time horizon, there is more room to derive a downward sloping aggregate saving function because of the wealth distribution among different generations (a comprehensive explanation of this channel can be found in Battacharya and Russell [5]). This mechanism is completely absent from the two-period model.

Furthermore, this new mechanism supports stable steady states for reasonable values of the discount factor. When the discount factor is relatively high, young households save almost all their corresponding endowment regardless of the rate of return, so their demand for savings becomes more inelastic to the rate of return. But since the new transmission channel is strong, indeterminacy persists for values of the discount factor of the order of one-half.

As explained above, indeterminacy means that in the neighborhood of a stable steady state there is an infinite number of convergent price sequences. This feature of forward stability will be used to construct stationary sunspot equilibria in the next section.

## Two-period economies

In this subsection I revisit briefly the model originally posed in Lahiri and Puhakka [20], inhabited by two-period-lived agents. If  $\alpha_t$  stands for the savings of the young, then in any competitive equilibrium the asset market clears when  $\alpha_{t+1} = R_t \alpha_t$ . Differentiating this expression I get  $\partial \alpha_{t+1} / \partial \alpha_t = R_t (1 + 1/\epsilon)$ , where  $\epsilon = (d\alpha/dR)(R/\alpha)$ . For indeterminacy to occur, I need  $\epsilon < -0.5$  at the steady state. As before, an economy can be represented by a

triplet  $(\omega_1, \gamma, \beta) \in [0, 1] \times [0, 1] \times (0, 1]$ . I depict in Figure 2 my results:

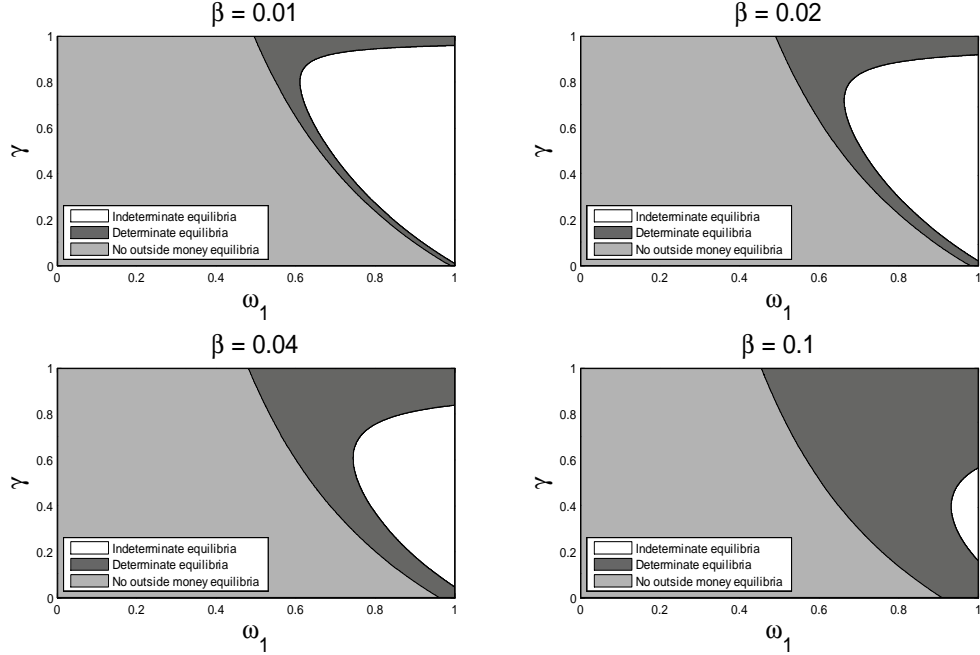


Figure 2. Stability properties of the steady state, for different parameter configurations  $(\omega_1, 1 - \omega_1, \gamma, \cdot)$  and  $\beta \in \{0.01, 0.02, 0.04, 0.1\}$ . At every steady state, *indeterminacy* occurs when the elasticity of savings with respect to the real rate of return, evaluated at the steady state, is less than minus one-half; *determinacy* occurs when the elasticity is greater than minus one-half and, finally, *no outside money equilibrium* means that a condition similar to Assumption 1 is not satisfied.

Unlike the case with 3 periods, now income should be heavily concentrated in the early parts of the life cycle and the discount factor should be very low for indeterminacy to occur. It is also evident that indeterminacy is not monotonic in the habit parameter. In order to study this phenomenon, I derive the offer curve a la Cass, Okuno and Zilcha [9] when  $(\omega_1, \omega_2, \gamma, \beta) = (0.95, 0.05, \cdot, 0.1)$  and  $\gamma \in \{0.1, 0.4, 0.7\}$ . The offer curve can be represented as  $\alpha_t = g(\alpha_{t+1})$ , using backward dynamics<sup>8</sup>.

<sup>8</sup>See the appendix for details on the derivation of the offer curve.

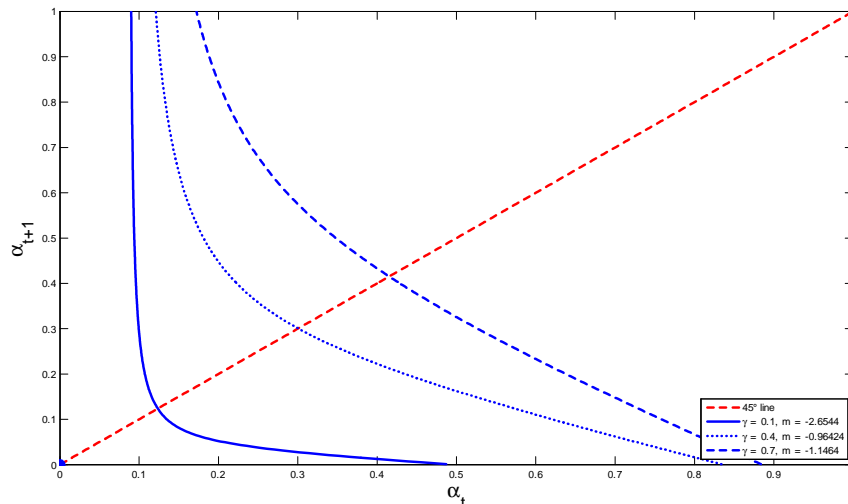


Figure 3. Offer curve associated to OLG economies with two-period-lived agents when  $(\omega_1, \omega_2, \beta) = (0.95, 0.05, 0.1)$ . The habit parameter  $\gamma$  takes three different values, and  $m$  stands for the value of the slope of the offer curve when it crosses the 45-degree line.

As shown in Figure 3, the offer curve is surjective (there are two values of  $\alpha_t$  consistent with  $\alpha_{t+1} = 0$ ), and hence discontinuous. The offer curve is negatively sloped and, most importantly, the slope of the curve at the steady state is less than one in absolute value when  $\gamma = 0.4$ , but greater than one when  $\gamma$  is either 0.1 or 0.7. Unlike the standard case with power utility and low elasticity of substitution, the steady state is stable even though the offer curve does not bend backwards (as stated incorrectly in Lahiri and Puhakka [20]). Since the higher the habit parameter, the flatter the offer curve, it is clear that indeterminacy may not be monotonic in the habit parameter.

With two periods, households only accumulate savings when young (if any). When the discount factor is low, they desire to consume in the first period of life. But in presence of habit formation, consumption when young increases the marginal utility of consumption when old. Consequently, their initial willingness to save is high, so an increase in the rate of return produces a non-negligible raise in the level of savings. However, the income effect acts in the opposite direction. If the net effect of these two forces is strong enough to produce a decrease in the level of savings, and the offer curve intersects the 45-degree line with a slope less than one in absolute value, then there is indeterminacy.

## 4 Rational expectations equilibrium

An alternative interpretation of Proposition 4 is that there is a set of economies in which neither preferences nor endowments may help determine the actual outcome at any period  $t$ . Unfortunately, habit formation may undermine the concept of perfect foresight equilibrium and in this scenario, agents rely on an extrinsic random variable, or sunspot variable, which conveys useful information for forecasting equilibrium prices.

To construct an equilibrium, following Spear, Srivastava and Woodford [24], I first study the problem faced by the agent in the last period of life (i.e. backward dynamic programming). I consider stochastic economies close to a deterministic economy satisfying Proposition 4. Loosely speaking, I will be looking for a stationary forecast process satisfying some optimality conditions, together with an invariant measure for the process defined by the forecast function, that is, the stochastic counterpart of Definition 2 (a formal definition is provided in Definition 3).

Let  $c_{3,t+2}[m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}]$  be the third period demand function obtained from:

$$\max_{c_{3,t+2}} U[c_{1,t}, c_{2,t+1}, c_{3,t+2}]$$

subject to

$$p_{t+2}c_{3,t+2} \leq p_{t+2}\omega_3 + m_{2,t+1}$$

Let  $\lambda_2[m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}]$  be the associated multiplier. Now, in the second period of life, the agent thinks of future prices as functions of  $\{\varepsilon_t\}$ , a sequence of bounded, independent and identically distributed random variables, with distribution given by the measure  $\mu$ . The agent also knows that  $\psi$  is the induced probability measure of period  $t+2$  prices. When middle-aged, the expected utility maximization problem is as follows:

$$\max_{c_{2,t+1}, m_{2,t+1}} \int U[c_{1,t}, c_{2,t+1}, c_{3,t+2}[m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}]] d\psi[p_{t+2}]$$

subject to

$$p_{t+1}c_{2,t+1} \leq p_{t+1}\omega_2 - m_{2,t+1} + m_{1,t}$$

The first-order conditions for this problem are:

$$\begin{aligned} \int D_2 U[c_{1,t}, c_{2,t+1}, c_{3,t+2}[m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}]] d\psi[p_{t+2}] - \lambda_1 p_{t+1} &= 0 \\ \lambda_1 - \int \lambda_2[m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}] d\psi[p_{t+2}] &= 0 \\ p_{t+1}z_{2,t+1} + m_{2,t+1} - m_{1,t} &= 0 \end{aligned} \quad (7)$$

where  $z_{2,t+1} = c_{2,t+1} - \omega_2$ . The implementation of a stationary sunspot equilibrium makes use of the bootstrapping method, which consists of replacing the stochastic first order

conditions with equations in which there is one deterministic component and one random component. Hence I rewrite the conditions as:

$$\begin{aligned}
D_2U [c_{1,t}, c_{2,t+1}, c_{3,t+2}[m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}]] - \lambda_1 p_{t+1} - \varepsilon_{c,t+2} &= 0 \\
\lambda_1 - \lambda_2 [m_{2,t+1}, c_{1,t}, c_{2,t+1}, p_{t+2}] - \varepsilon_{\lambda,t+2} &= 0 \\
p_{t+1} z_{2,t+1} + m_{2,t+1} - m_{1,t} &= 0
\end{aligned} \tag{8}$$

where  $\varepsilon_{t+2} = (\varepsilon_{c,t+2}, \varepsilon_{\lambda,t+2})$  is an  $\mathbb{R}^2$ -valued random variable drawn independently each period from a distribution  $\mu$  such that  $\int \varepsilon d\mu [\varepsilon] = 0$ . Equations (8) imply (7). Now I substitute from the budget constraint for  $c_{2,t+1}$  in the first two equations. Then I substitute from the multiplier equations for  $\lambda_1$  in the first equation. This yields:

$$\begin{aligned}
D_2U [c_{1,t}, m_{2,t+1}, c_{3,t+2}[m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, p_{t+2}]] \\
- \lambda_2 [m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, p_{t+2}] p_{t+1} - \varepsilon_{\lambda,t+2} p_{t+1} - \varepsilon_{c,t+2} &= 0
\end{aligned} \tag{9}$$

Note that with  $m_{1,t}$  and  $c_{1,t}$  given,  $p_{t+1} = p_{t+2} = p$ , and  $\varepsilon_{t+2} = 0$ , this equation will have a solution  $m_2$ . Now I wish to solve this equation for  $p_{t+2}$  as a function of  $(m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, \varepsilon_{t+2})$  on a neighborhood of  $(m_{1,t}, c_{1,t}, m_2, p, 0)$ . In order to apply the implicit function theorem, I need the derivative:

$$J = D_{23}UD_{p_{t+2}}c_{3,t+2} - p_{t+1}D_{p_{t+2}}\lambda_2$$

to be non-zero. For generic utility functions, Spear, Srivastava and Woodford [24] argue that this is the case.

**Proposition 5** *For an open and dense set of utility functions satisfying the maintained assumptions,  $J$  is non-zero at  $(m_{1,t}, c_{1,t}, m_2, p, 0)$ .*

**Proof.** See Spear, Srivastava and Woodford [24, Lemma 4.1]. ■

Given this result, it is possible to write:

$$p_{t+2} = H_1 [m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, \varepsilon_{t+2}] \tag{10}$$

such that  $H_1$ ,  $D_3H_1$  and  $D_4H_1$  are all continuous functions of  $(m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, \varepsilon_{t+2})$ . As in Spear [23], I assume that the realization of  $\varepsilon_{t+2}$  and determination of  $p_{t+2}$  during period  $t+2$  are simultaneous. Now the agent takes this forecast function as given, together with the distribution  $\mu$  for  $\varepsilon$  and generate demands  $c_{2,t+1} [m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}; \mu]$  and shadow prices  $\lambda_1 [m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}; \mu]$ .

In the first period of life, the agent considers the following expected utility maximization



problem:

$$\max_{c_{1,t}, m_{1,t}} \int U [c_{1,t}, c_{2,t+1} [m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}; \mu], c_{3,t+2} [m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, H_1; \mu]] d\psi [p_{t+1}]$$

subject to

$$p_t c_{1,t} + m_{1,t} \leq p_t \omega_1$$

The first-order conditions for this problem are:

$$\begin{aligned} \int D_1 U [c_{1,t}, c_{2,t+1} [\circ], c_{3,t+2} [\bullet]] d\psi [p_{t+1}] - \lambda_0 p_t &= 0 \\ \lambda_0 - \int \lambda_1 [\circ] d\psi [p_{t+1}] &= 0 \\ p_t z_{1,t} + m_{1,t} &= 0 \end{aligned} \tag{11}$$

where  $z_{1,t} = c_{1,t} - \omega_1$ ,  $(\circ) = (m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}; \mu)$  and  $(\bullet) = (m_{1,t}, c_{1,t}, m_{2,t+1}, p_{t+1}, H_1; \mu)$ . It is worth mentioning that  $H_1$  does not depend on  $\varepsilon_{t+2}$ , as it has been integrated out. Now I replace the above equations by:

$$\begin{aligned} D_1 U [c_{1,t}, c_{2,t+1} [\circ], c_{3,t+2} [\bullet]] - \lambda_0 p_t - \varepsilon_{c,t+1} &= 0 \\ \lambda_0 - \lambda_1 [\circ] - \varepsilon_{\lambda,t+1} &= 0 \\ p_t z_{1,t} + m_{1,t} &= 0 \end{aligned} \tag{12}$$

Equations (12) imply (11). As before, I substitute from the budget constraint for  $c_{1,t}$  in the first two equations. Next I substitute from the multiplier equations for  $\lambda_0$  in the first equation. This yields:

$$\begin{aligned} D_1 U [m_{1,t}, c_{2,t+1} [m_{1,t}, m_{2,t+1}, p_t, p_{t+1}; \mu], c_{3,t+2} [m_{1,t}, m_{2,t+1}, p_t, p_{t+1}, H_1 [m_{1,t}, m_{2,t+1}, p_t, p_{t+1}]; \mu]] \\ - \lambda_1 [m_{1,t}, m_{2,t+1}, p_t, p_{t+1}; \mu] p_t - \varepsilon_{\lambda,t+1} p_t - \varepsilon_{c,t+1} = 0 \end{aligned} \tag{13}$$

Note that at  $p_t = p_{t+1} = p$ , and  $\varepsilon_{t+1} = 0$ , this equation will have a solution  $m_1$ . Now I wish to solve this equation for  $p_{t+1}$  as a function of  $(m_{1,t}, m_{2,t+1}, p_t, \varepsilon_{t+1})$  on a neighborhood of  $(m_1, m_2, p, 0)$ . Let  $\eta^{t+1} = (m_{1,t}, m_{2,t+1})$ . As in the previous case, in order to apply the implicit function theorem I need the derivative:

$$J' = D_{12} U D_{p_{t+1}} c_{2,t+1} + D_{13} U (D_{p_{t+1}} c_{3,t+2} + D_{p_{t+2}} c_{3,t+2} D_{p_{t+1}} H_1 [\eta^{t+1}, p_t, p_{t+1}]) - p_t D_{p_{t+1}} \lambda_1$$

to be non-zero. Hence by an argument similar to that used above, it can be shown that  $J'$  is also non-zero generically. Then this equation can be solved for  $p_{t+1}$  on a neighborhood of the non-stochastic steady state values  $p$ ,  $\eta$ , and  $\varepsilon_{t+1} = 0$ . Let the one-period-ahead forecast

be given by:

$$p_{t+1} = H_1' [\eta^{t+1}, p_t, \varepsilon_{t+1}] \quad (14)$$

When I plug (14) into (10), and incorporate the relevant budget constraint, I get the two-period-ahead forecast, which is the relevant forecast function:

$$p_{t+2} = H_2 [\eta^{t+1}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}] \quad (15)$$

Given this result, there exist neighborhoods  $\mathbf{M}_1$  of  $\eta$ ,  $\mathbf{P}_0$ ,  $\mathbf{P}_1$  of  $p$ , and  $\mathbf{N}_1$  of 0 such that (15) has a unique solution  $p_{t+2} \in \mathbf{P}_0$  for each choice of  $\eta^{t+1} \in \mathbf{M}_1$ ,  $p_t \in \mathbf{P}_1$ , as well as  $\varepsilon_{t+1}$  and  $\varepsilon_{t+2} \in \mathbf{N}_1$ . Now let agents take this forecast function as given, together with the distribution  $\mu$  for  $\varepsilon$  and generate demands  $c_{1,t} [\eta^{t+1}, p_t; \mu]$  and shadow prices  $\lambda_0 [\eta^{t+1}, p_t; \mu]$ .

Finally, since  $m_{1,t} = -p_t z_{1,t}$  and  $m_{2,t+1} = -p_{t+1} z_{2,t+1} - p_t z_{1,t}$ , the market-clearing condition at  $t + 1$  reads:

$$m_{1,t+1} [\eta^{t+1}, \eta^{t+2}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}] + m_{2,t+1} [\eta^{t+1}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}] - \bar{m} = 0 \quad (16)$$

Hence by an argument similar to that used before to establish that  $J$  and  $J'$  are generically non-zero, it can be shown that this equation can be solved generically for  $\eta^{t+2}$  on a neighborhood of the non-stochastic steady state values  $p$ ,  $\eta$ ,  $\varepsilon = 0$  and  $\mu = \bar{\mu}$ . Formally I obtain the following result:

**Proposition 6** *For an open and dense set of utility functions, equations (9), (13) and (16) can be solved for  $\eta^{t+2}$ . Specifically for any function in that set, there exist neighborhoods  $\mathbf{P}_2 \subset \mathbf{P}_1$  of  $p$ ,  $\mathbf{M}_0$  and  $\mathbf{M}_2 \subset \mathbf{M}_1$  of  $\eta$ , and  $\mathbf{N}_2 \subset \mathbf{N}_1$  of 0, and an  $\zeta > 0$ , such that (9), (13) and (16) have a unique solution  $\eta^{t+2} \in \mathbf{M}_0$ , for each choice of  $p_t \in \mathbf{P}_2$ ,  $\varepsilon_{t+1}, \varepsilon_{t+2} \in \mathbf{N}_2$ ,  $\eta^{t+1} \in \mathbf{M}_2$ , and for any choice of  $\mu$  such that  $\|\varepsilon\| < \zeta$  with probability one. Furthermore, the solution*

$$\eta^{t+2} = G_1 [\eta^{t+1}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}]$$

*is such that  $G_1$ ,  $D_1 G_1$  and  $D_2 G_1$  are all continuous functions of  $\eta^{t+1}$ ,  $p$ ,  $\varepsilon_{t+1}$  and  $\varepsilon_{t+2}$ .*

**Proof.** See Spear, Srivastava and Woodford [24, Lemma 4.2]. ■

In order to complete the characterization of the equilibrium, I consider again the forecast and allocations functions. Clearly there are lagged allocations (allocation state variables) in the right hand side of both functions:

$$\begin{aligned} p_{t+2} &= H_2 [\eta^{t+1}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}] \\ \eta^{t+2} &= G_1 [\eta^{t+1}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}] \end{aligned}$$

In what follows, I will show that it is convenient to express the forecast function solely in terms of lagged prices and random variables. As in Spear, Srivastava and Woodford [24], I invert the forecast function and solve for a function:

$$\eta^{t+1} = H_3 [p_{t+2}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}]$$

When I substitute this expression into the expression for  $\eta^{t+2}$  (the allocation function) I get:

$$\eta^{t+2} = G_2 [p_{t+2}, p_t, \varepsilon_{t+1}, \varepsilon_{t+2}]$$

Lagging this one period gives:

$$\eta^{t+1} = G_2 [p_{t+1}, p_{t-1}, \varepsilon_t, \varepsilon_{t+1}]$$

Substituting for  $\eta^{t+1}$  in the forecast function  $H_2$  then yields:

$$p_{t+2} = H_4 [p_{t+1}, p_t, p_{t-1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}] \tag{17}$$

which is a continuous function in the first five arguments and a measurable function of the fifth argument. Proposition 4 implies that in the case of  $\varepsilon_t = \varepsilon_{t+1} = \varepsilon_{t+2} = 0$ ,  $\mu = \bar{\mu}$  (the measure that puts all mass on  $\varepsilon_t = \varepsilon_{t+1} = \varepsilon_{t+2} = 0$ ), the non-stochastic dynamics are contracting in a neighborhood of the steady state  $p$ . By continuity,  $H_4$  is a contraction for  $\varepsilon_t$ ,  $\varepsilon_{t+1}$  and  $\varepsilon_{t+2}$  sufficiently close to zero and  $\mu$  sufficiently close to  $\bar{\mu}$ .

At this point it is useful to define the rational expectations equilibrium.

**Definition 3** *A stationary rational expectations equilibrium is a pair  $(H_4, \eta)$  such that  $H_4$  is a stationary forecast process satisfying the stochastic first-order conditions and the market-clearing condition and  $\eta$  is an invariant measure for the Markov process defined by  $H_4$ . The rational expectations equilibrium is non-trivially stochastic if the function  $H_4$  depends non-trivially on the random variable  $\varepsilon_{t+2}$ .*

As before, the third-order difference equation defined by  $H_4$  is converted into a first-order system of the form:

$$\begin{pmatrix} p_{t+2} \\ p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} H_4 [p_{t+1}, p_t, p_{t-1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}] \\ p_{t+1} \\ p_t \end{pmatrix} = \hat{H}_4 [p_{t+1}, p_t, p_{t-1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}]$$

or, alternatively,  $q_{t+2} = \hat{H}_4 [q_{t+1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}]$ , where  $q_{t+1} = (p_{t+1}, p_t, p_{t-1})'$ . Then,  $\hat{H}_4$  together with the measure  $\mu$  describe a stationary Markov process for  $(q_{t+1}, \varepsilon_t, \varepsilon_{t+1})$  with

support  $\mathbf{P}_3(p)^3 \times \mathbf{N}_3(0)^2$ , with  $\mathbf{P}_3 \subset \mathbf{P}_2 \cap \mathbf{P}_0$  and  $\mathbf{N}_3 \subset \mathbf{N}_2$ . Since  $\hat{H}_4$  is a contraction, it may well be assumed that this support is compact. Now it is left to show that there is an invariant measure  $\eta$  for this process.

As in Spear, Srivastava and Woodford [24], I consider any continuous function  $Q [q_{t+2}, \varepsilon_{t+1}, \varepsilon_{t+2}]$  and define the transition probability operator  $T$  associated with the measure on  $\mu$ :

$$(TQ) [q_{t+1}, \varepsilon_t, \varepsilon_{t+1}] = \int_{\mathbf{N}_3(0)} Q \left[ \hat{H}_4 [q_{t+1}, \varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}], \varepsilon_{t+1}, \varepsilon_{t+2} \right] d\mu [\varepsilon_{t+2}]$$

As  $\hat{H}_4$  is continuous in its first three arguments, the transition operator  $T$  (which is applied on the function  $Q$ ) plainly takes continuous functions into continuous functions. Hence, by Rosenblatt [21, Theorem IV.3.1], there exists an invariant measure for the stationary Markov process defined by  $\hat{H}_4$  and the measure  $\mu$ .

It turns out that the pair  $(H_4, \eta)$  stands for a rational expectations equilibrium. Since the random variable  $\varepsilon_{t+2}$  is non-degenerate by assumption, the agents' forecasts are non-trivially stochastic. Thus I have proved the following proposition:

**Proposition 7** *Under Proposition 4, there exists a non-empty, open set of economies exhibiting non-trivial stationary rational expectations equilibria  $(H_4, \eta)$  in state variables  $(p_{t+1}, p_t, p_{t-1}, \varepsilon_t, \varepsilon_{t+1})$  for  $\mu$  sufficiently close to  $\bar{\mu}$ .*

Therefore, in principle there is no causal connection between the sunspots and prices, except through agents' forecasts, an argument originally put forward by Spear [23].

## Two-period economies revisited

In the general multiple commodity model or in three-period-lived models, the allocation variables cannot be eliminated through the budget constraints and hence play a non-trivial role in determining the form of the equilibrium. However, in two-period-lived models it is possible to eliminate the allocations from the equilibrium price function via the money market clearing and the two budget constraints.

The proof is straightforward. First of all, the money market clearing condition requires  $m_t = \bar{m}$  for all  $t$ , and thus the sequential budget constraints may be rewritten as:

$$c_{1,t} \leq \omega_1 - \frac{\bar{m}}{p_t}, \text{ and } c_{2,t+1} \leq \omega_2 + \frac{\bar{m}}{p_{t+1}}$$

The first-order condition which define the equilibrium in this model then takes the form:

$$-\frac{1}{p_t} D_1 U \left[ \omega_1 - \frac{\bar{m}}{p_t}, \omega_2 + \frac{\bar{m}}{p_{t+1}} \right] + \frac{1}{p_{t+1}} D_2 U \left[ \omega_1 - \frac{\bar{m}}{p_t}, \omega_2 + \frac{\bar{m}}{p_{t+1}} \right] - \varepsilon_{t+1} = 0$$

This equation can be readily solved for  $p_{t+1} = F(p_t, \varepsilon_{t+1})$ . Under conditions similar to Proposition 4, the forecast function is consistent with stationary sunspot equilibria.

## 5 Concluding remarks

I show that indeterminacy may arise in three-period-lived OLG economies with reasonable values of the discount factor and hump-shaped endowment profiles even if the utility function is logarithmic, provided that subtractive habit formation is allowed for. When the steady state becomes locally stable, it is possible to construct sunspot equilibria.

It would be straightforward to assume that  $\omega_3$  is positive (but relatively small), in which case the results presented in this paper would hold by means of a continuity argument. Furthermore, if I relied on more general power utility functions beyond the logarithmic case, I would obtain local stability with a higher elasticity of substitution than what is needed in traditional models (see Grandmont [13]). A larger number of periods would just reinforce the income effect.

The approach presented in this paper complements the huge literature on general equilibrium that argues that indeterminacy helps arbitrary self-fulfilling beliefs become a source of volatility, even when preferences or endowments are not random. Even though the literature on extrinsic uncertainty has always highlighted the role of sunspot variables as coordination devices, the theory developed here is unable to identify the way in which market participants coordinate their expectations and consequent decisions. In other settings, however, it is possible to grasp such relationship, for example, Duffy and Fisher [10] present direct evidence from a controlled experiment on how buyers and sellers in a double action use a sunspot variable as a coordination mechanism. I believe this is a promising direction of research.

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## Appendix

### A Proofs

**Proof of proposition 1.** Note that  $m_{1,t} [p_{t+2}, p_{t+1}, p_t] = p_t \omega_1 - p_t c_{1,t}$ , and  $m_{2,t} [p_{t+1}, p_t, p_{t-1}] = p_{t-1} \omega_1 + p_t \omega_2 - (p_{t-1} c_{1,t-1} + p_t c_{2,t})$ , where  $c_{1,t}$  and  $c_{2,t+1}$  are defined as (4) and (5), respectively.

The market-clearing condition can be written as:

$$\begin{aligned} m_{1,t} [p_{t+2}, p_{t+1}, p_t] &= \bar{m} - m_{2,t} [p_{t+1}, p_t, p_{t-1}] \\ p_t \omega_1 - \frac{p_t (p_t \omega_1 + p_{t+1} \omega_2)}{(1 + \beta + \beta^2) (p_t + \gamma p_{t+1} + \gamma^2 p_{t+2})} &= \bar{m} - m_{2,t} [p_{t+1}, p_t, p_{t-1}] \end{aligned}$$

Or, alternatively:

$$p_{t+2} = \frac{1}{\gamma^2} \left\{ \frac{\delta p_t (p_t \omega_1 + p_{t+1} \omega_2)}{p_t \omega_1 + m_{2,t} [p_{t+1}, p_t, p_{t-1}] - \bar{m}} - (p_t + \gamma p_{t+1}) \right\}$$

where  $\delta = (1 + \beta + \beta^2)^{-1}$ . Clearly, this last expression can be written as  $p_{t+2} = F_1 [p_{t+1}, p_t, p_{t-1}]$ .

■

**Proof of proposition 2.** At any stationary equilibrium,  $p_t = p$ . Then,

$$m_1 [p] = p (\omega_1 - c_1)$$

where  $c_1 = (\omega_1 + \omega_2) ((1 + \beta + \beta^2) (1 + \gamma + \gamma^2))^{-1}$ . In the same fashion,

$$m_2 [p] = p ((\omega_1 + \omega_2) - (1 + (1 + \beta)\gamma + \beta / (1 + \gamma)) c_1)$$

The market clearing condition is  $m_1 [p] + m_2 [p] = \bar{m}$ , or:

$$p (\omega_1 (2 - \lambda) + \omega_2 (1 - \lambda)) = \bar{m}$$

where  $\lambda = \frac{(1+\beta)(1+\gamma+\gamma^2)+1+2\gamma}{(1+\beta+\beta^2)(1+\gamma+\gamma^2)(1+\gamma)}$ .

Solving out for  $p$  yields:

$$p = (\omega_1 (2 - \lambda) + \omega_2 (1 - \lambda))^{-1} \bar{m}$$

Clearly, the steady-state price is positive if either  $\bar{m} > 0$  and  $(\omega_1 (2 - \lambda) + \omega_2 (1 - \lambda)) > 0$ , or  $\bar{m} < 0$  and  $(\omega_1 (2 - \lambda) + \omega_2 (1 - \lambda)) < 0$ . ■



## B Derivation of offer curve

Let  $R_t = p_t/p_{t+1}$ . The excess demand function when old is:

$$\begin{aligned}
 \alpha_{t+1} &= c_{2,t+1} - \omega_2 \\
 &= \omega_1 R_t - c_{1,t} R_t \\
 &= \omega_1 R_t - \frac{\omega_1 R_t^2 + \omega_2 R_t}{(1 + \beta)(R_t + \gamma)} \\
 &= \frac{R_t [\gamma \omega_1 - \omega_2 + \beta \omega_1 (R_t + \gamma)]}{(1 + \beta)(R_t + \gamma)}
 \end{aligned}$$

where the second and third lines use  $\alpha_{t+1} = R_t \alpha_t$  and  $c_{1,t} = (\omega_1 R_t + \omega_2)(1 + \beta)^{-1}(R_t + \gamma)^{-1}$ , respectively. The last expression can be written as:

$$0 = \lambda_2 R_t^2 + \lambda_1 R_t + \lambda_0$$

where  $\lambda_2 = \beta \omega_1$ ,  $\lambda_1 = (1 + \beta) \gamma \omega_1 - \omega_2 - (1 + \beta) \alpha_{t+1}$ , and  $\lambda_0 = -(1 + \beta) \gamma \alpha_{t+1}$ . From this quadratic form I get:

$$\tilde{R} = \frac{-\lambda_1 \pm (\lambda_1^2 - 4\lambda_2\lambda_0)^{\frac{1}{2}}}{2\lambda_2}$$

By Descartes's rule of signs, there is only one positive root  $\tilde{R}^+$  (because  $\lambda_2 > 0$ ,  $\lambda_0 < 0$  and  $\lambda_1 \neq 0$ ). Finally, the level of savings is given by:

$$\alpha_t = \frac{\omega_1 [\gamma + \beta (\tilde{R}^+ + \gamma)] - \omega_2}{(1 + \beta) (\tilde{R}^+ + \gamma)}$$

The offer curve is the locus of points  $(\alpha_t, \alpha_{t+1})$ .